

AKEMANN - OSTRAND PROPERTY FOR $\mathrm{PGL}_2(\mathbb{Z}[\frac{1}{p}])$ RELATIVE TO $\mathrm{PSL}_2(\mathbb{Z})$

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ABSTRACT. We generalize the Akemann - Ostrand theorem for $\mathrm{PSL}_2(\mathbb{Z})$, to the case of the partial transformations action of $\mathrm{PGL}_2(\mathbb{Z}[\frac{1}{p}]) \times \mathrm{PGL}_2(\mathbb{Z}[\frac{1}{p}])^{\mathrm{op}}$, by left and and right multiplication on $\mathrm{PSL}_2(\mathbb{Z})$.

0. INTRODUCTION

In this paper we consider a generalization of the Akemann - Ostrand theorem ([1]) for $\mathrm{PSL}_2(\mathbb{Z})$, to the case of the partial transformations action of

$$\mathrm{PGL}_2(\mathbb{Z}[\frac{1}{p}]) \times \mathrm{PGL}_2(\mathbb{Z}[\frac{1}{p}])^{\mathrm{op}},$$

by left and and right multiplication on $\mathrm{PSL}_2(\mathbb{Z})$.

Recall that Akemann - Ostrand property (to which we will refer in the sequel as to the AO property) for the free group F_N , $N \geq 2$ asserts ([1]) the fact that the C^* - algebra, generated in $\mathcal{B}(l^2(F_N))$, simultaneously by the C^* - algebras $C_\lambda^*(F_N)$, $C_\rho^*(F_N)$ that are generated by the left and respectively, the right convolution operators with elements in F_N , is isomorphic, modulo the ideal $\mathcal{K}(l^2(F_N))$ of compact operators, to the minimal C^* - tensor product $C_{\mathrm{red}}^*(F_N) \otimes_{\min} C_{\mathrm{red}}^*(F_N^{\mathrm{op}}) \cong C_{\mathrm{red}}^*(F_N \times F_N^{\mathrm{op}})$ of the reduced group C^* - algebras

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associated to F_N . Here, by definition the reduced group C^* - algebra $C_{\text{red}}^*(\Gamma)$ of a discrete group Γ is $C_\lambda^*(\Gamma)$.

The Akemann - Ostrand property has been widely extended. G. Skandalis proved ([31]) that the same result remains true for lattices in semisimple Lie groups of rank 1. Using amenable actions techniques ([2]), Gunter and Higson ([12]) and then Ozawa ([24]) have further extended this result, to large classes of hyperbolic groups.

The key in Ozawa's approach in proving the AO property for a discrete group Γ is the amenability ([35], [3], [4], [14]) of the action of $\Gamma \times \Gamma^{\text{op}}$ on the boundary $\partial(\beta\Gamma)$ of the Stone Cech compactification ([7]) of Γ , viewed as a discrete set. This stronger property for a group Γ is called ([24],[5]) property \mathcal{S} .

Consider the canonical representation, which we denote by π_{Koop} , of the crossed product C^* - algebra $C^*((\Gamma \times \Gamma^{\text{op}}) \ltimes C(\partial(\beta\Gamma)))$ into $\mathcal{B}(l^2(\Gamma))$. Let

$$\pi_{\text{Calk}} : \mathcal{B}(l^2(\Gamma)) \rightarrow \mathcal{Q}(l^2(\Gamma)) = \mathcal{B}(l^2(\Gamma)) / \mathcal{K}(l^2(\Gamma))$$

be the projection onto the Calkin algebra. The \mathcal{S} property ([23]) implies that the representation

$$\pi_{\text{Calk}} \circ \pi_{\text{Koop}} : C^*((\Gamma \times \Gamma^{\text{op}}) \ltimes C(\partial(\beta\Gamma))) \rightarrow \mathcal{Q}(l^2(\Gamma)),$$

factorizes to a representation of the reduced C^* - algebra

$$C_{\text{red}}^*((\Gamma \times \Gamma^{\text{op}}) \ltimes C(\partial(\beta\Gamma))).$$

In this paper we extend the Akemann - Ostrand property in the following sense. Let Γ be the modular group $\text{PSL}_2(\mathbb{Z})$. Let $G = \text{PGL}_2(\mathbb{Z}[\frac{1}{p}])$, p a prime ≥ 2 . It is well known ([11]), that Γ is almost normal in G . The almost normal property for the subgroup Γ of G signifies that for all $g \in G$ the subgroup

$$(1) \quad \Gamma_g = g\Gamma g^{-1} \cap \Gamma \subseteq \Gamma,$$

has finite index $[\Gamma : \Gamma_g]$.

The group G acts naturally, by conjugation, by partial isomorphisms, on Γ . Indeed for $g \in G$, the conjugation by g on G , will restrict to a partial isomorphism

$$(2) \quad \Delta(g) : \Gamma_{g^{-1}} \rightarrow \Gamma_g.$$

It is well known that in the case of the example of the modular group, that we are considering in this paper, we have that $[\Gamma : \Gamma_g] = [\Gamma : \Gamma_{g^{-1}}]$, for all $g \in G$. We consider the family of maximal normal subgroups Γ_g^0 contained in Γ_g . Clearly, if $s_i^g, i = 1, 2, \dots, [\Gamma : \Gamma_g]$, are the left cosets representatives for the subgroup Γ_g in Γ , then

$$\Gamma_g^0 = \bigcap_{i=1}^{[\Gamma:\Gamma_g]} s_i \Gamma_g s_i^{-1}.$$

Let K be the compact space obtained as the inverse limit of the finite coset spaces Γ/Γ_g^0 as $g \rightarrow \infty$. Then K is a totally disconnected subgroup, with Haar measure μ_K defined by the requirement that the compact set corresponding to the closure of a coset $s\Gamma_g, s \in \Gamma$ in the profinite topology, has Haar measure equal to $\frac{1}{[\Gamma : \Gamma_g]}, g \in G$.

The condition that $[\Gamma : \Gamma_g] = [\Gamma : \Gamma_{g^{-1}}]$, implies that the partial transformation $\Delta(g)$, introduced in formula 3 induced by conjugation with $g \in G$, preserves the Haar measure μ_K on K .

There is a natural action of $G \times G^{\text{op}}$ on K . An element $(g_1, g_2) \in G \times G^{\text{op}}$ acts by partial transformations on K , by mapping, $k \in K$ into $g_1 k g_2^{-1}$, if the later element also belongs to K . Thus, the domain of (g_1, g_2) , as a partial transformation on K , is

$$\mathcal{D}_{(g_1, g_2)} = \{k \in K \mid g_1 k g_2^{-1} \in K\} = K \cap g_1^{-1} K g_2 = K \cap g_1^{-1} K g_1 (g_1^{-1} g_2).$$

We use the notation $K_g = K \cap gKg^{-1}$. In our construction this is the closure, in the profinite completion, of Γ_g . Then

$$\mathcal{D}_{(g_1, g_2)} = (K_{g_1^{-1}})g_1^{-1}g_2,$$

which is a coset of $K_{g_1^{-1}}$.

The above introduced transformation group is used to define the groupoid crossed product C^* - algebra

$$C^*((G \times G^{\text{op}}) \ltimes C(K)).$$

Since each element in $G \times G^{\text{op}}$ acts by measure preserving transformation on its domain it follows that the measure μ_K on K induces a trace τ on $C^*((G \times G^{\text{op}}) \ltimes C(K))$.

Then τ can be used to define the reduced C^* - algebra of the groupoid crossed product

$$C_{\text{red}}^*((G \times G^{\text{op}}) \ltimes C(K)).$$

It is well known ([21],[32]) that, in order to construct the C^* -algebra above, one may use the G.N.S. representation associated to τ .

Denote the identity element of G by 1. The crossed product C^* -algebra $C^*((G \times G^{\text{op}}) \ltimes C(K))$ may also be realized as the groupoid crossed product algebra

$$C^*(\Delta(G) \ltimes (C^*((\Gamma \times 1^{\text{op}}) \ltimes C(K)))).$$

This is because the action of each $\Delta(g)$, $g \in G$, by partial isomorphisms on $C(K)$, may be canonically extended to a partial transformations action on $C^*((\Gamma \times 1^{\text{op}}) \ltimes C(K))$.

We denote this extension also by $\Delta(g)$. The extension is defined by the requirement that it maps

$$\gamma \in \Gamma_g \cong \Gamma_g \times 1^{\text{op}} \subseteq \Gamma \times 1^{\text{op}}$$

into the transformation

$$(3) \quad \Delta(g)(\gamma) = g\gamma g^{-1}, \gamma \in \Gamma_g.$$

The Haar measure μ_K on K induces a trace on $C^*((\Gamma \times 1^{\text{op}}) \ltimes C(K))$, which is the restriction of the trace τ introduced above. The trace τ is invariant to the action of the transformations $\Delta(g)$, $g \in G$.

Consequently we have the isomorphism

$$(4) \quad C_{\text{red}}^*(\Delta(G) \ltimes C_{\text{red}}^*((\Gamma \times 1^{\text{op}}) \ltimes C(K))) \cong C_{\text{red}}^*((G \times G^{\text{op}}) \ltimes C(K)).$$

The C^* -algebra $C^*((G \times G^{\text{op}}) \ltimes C(K))$ is generated by $C^*((\Gamma \times \Gamma^{\text{op}}) \ltimes C(K))$ and $\Delta(\sigma_p)$, where

$$(5) \quad \sigma_p = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}.$$

Indeed

$$(G \times G^{\text{op}}) = (\Gamma \times 1)\Delta(G)(\Gamma \times 1).$$

The algebra $C(K)$ admits a canonical identification with a closed subalgebra of $\ell^\infty(G)$. This is realized by identifying the characteristic function χ_{sK_g} , for $s \in \Gamma$, of a coset of a subgroup K_g with the characteristic function $\chi_{s\Gamma_g} \in \ell^\infty(\Gamma)$. Consequently $C(K)$ embeds into $\ell^\infty(\Gamma)$, and the embedding is $G \times G^{\text{op}}$ -equivariant.

Hence we get a canonical representation π_{Koop} of the reduced crossed product C^* -algebra $C^*((G \times G^{\text{op}}) \ltimes C(K))$ into $\mathcal{B}(\ell^2(\Gamma))$.

We prove the following theorem

Theorem 1. *Let π_{Calk} , π_{Koop} be the representations introduced above. Then $\pi_{\text{Calk}} \circ \pi_{\text{Koop}}$ induces an isomorphism of the reduced C^* -algebra $C_{\text{red}}^*((G \times G^{\text{op}}) \ltimes C(K))$ into the Calkin algebra $\mathcal{Q}(\ell^2(\Gamma))$.*

If one restricts $\pi_{\text{Calk}} \circ \pi_{\text{Koop}}$ to the C^* -algebra $C^*((\Gamma \times \Gamma^{\text{op}}) \ltimes C(K))$, the isomorphism property this follows from the Ozawa's proof ([23]) of the Akemann - Ostrand property.

To prove the theorem, we will use the formalism of Loeb measure spaces. This is because when looking at states on $C^*((G \times G^{\text{op}}) \ltimes C(K))$ obtained by composing states on the Calkin algebra, with $\pi_{\text{Calk}} \circ \pi_{\text{Koop}}$, one is led to consider, because of Calkin, states that on $(g_1, g_2) \in G \times G^{\text{op}}$ are of the form

$$\phi_{\omega, A}(g_1, g_2) = \lim_{n \rightarrow \omega} \frac{\text{card}(g_1 A_n g_2 \cap A_n)}{\text{card} A_n}, g_1, g_2 \in G.$$

The above states are easily interpreted as matrix coefficients with respect to the Koopmann measure for a unitary representation into an infinite measure space (\mathcal{Y}, ν) , where ν is a σ -finite, $G \times G^{\text{op}}$ invariant measure.

The states $\phi_{\omega, A}$ on $C^*((G \times G^{\text{op}}) \ltimes \ell^\infty(\Gamma))$ do not necessary come from a Koopmann representation. Constructing the $G \times G^{\text{op}}$ -equivariant lifting to \mathcal{Y} which is a subspace of the non-standard universe ${}^*\Gamma$ ([27]), endowed with a suitable Loeb uniform counting measure ([16]) proves that the GNS construction associated with $\phi_{\omega, A}$ is embeddable in a Koopman representation which is easier to analyze.

The space \mathcal{Y} is constructed as a measurable fibration over $\partial(\beta(G))$. The fiber over a character ε on $C(\beta(G))$, keeps track on all possible ways to obtain ε , as a limit, over the ultrafilter ω , of sequences in Γ .

Using a this we prove that the crossed product C^* - algebra

$$C^*((\Gamma \times 1) \times L^\infty(\mathcal{Y}, \nu)),$$

in its Koopmann representation $\pi_{\mathcal{Y}}$ on $L^2(\mathcal{Y}, \nu)$ is a nuclear C^* -algebra.

The group $\Delta(G)$ acts on the center of the von Neumann algebra generated by the above algebra. We prove that this action is amenable.

Hence $C^*(\Delta(G) \ltimes C^*(\Gamma \times L^\infty(\mathcal{Y}, \nu)))$ is nuclear. Hence, by restricting the Koopman representation $\pi_{\mathcal{Y}}$ to the C^* -algebra

$$C^*(\Delta(G) \ltimes C^*((\Gamma \times 1^{\text{op}}) \ltimes L^\infty(\Gamma))),$$

we prove the theorem.

The result in this paper is the second part of a circulated preprint which has been rewritten to ease the understanding of the the proof of the result.

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1. OUTLINE OF THE PROOF

By Calkin results [Ca] to determine the norm on the Calkin algebra $\mathcal{Q}(H) = \mathcal{B}(H)/\mathcal{K}(H)$ associated to a Hilbert space, it is sufficient to consider an ultrafilter ω , and to consider the states on $\mathcal{B}(H)$ defined by the formula

$$(6) \quad \lim_{n \rightarrow \omega} \langle \cdot, \xi_n, \xi_n \rangle,$$

where the sequence $(\xi_n)_{n \in \mathbb{N}}$ runs over all sequences of norm 1 vectors in H , that are weakly convergent to 0.

The states we are considering in order to prove Theorem 1 are obtained by composing the states in formula 6 with have the representation π_{Koop} of $C^*((G \times G^{\text{op}}) \ltimes C(K))$ into $\mathcal{B}(l^2(\Gamma))$. To prove the theorem it is sufficient to

consider sequences of vectors $(\xi_n)_n$ on Γ , that, as functions on Γ are finitely supported, and have positive values on Γ .

The determination of the C^* norm on the crossed product algebra $C^*((G \times G^{\text{op}}) \ltimes C(K))$, in the above representation, consequently comes to determining state on $C^*((G \times G^{\text{op}}) \ltimes K)$, which on $(g_1, g_2) \in G \times G^{\text{op}}$ takes the value

$$\lim_{n \rightarrow \omega} \langle g_1 \chi_{\mathcal{D}(g_1, g_2)} \xi_n g_2, \xi_n \rangle,$$

where $\chi_{\mathcal{D}(g_1, g_2)}$ is the characteristic function of the domain of $(g_1, g_2) \in G \times G^{\text{op}}$ acting on K .

The vectors $(\xi_n)_n$ also determine a positive state on $l^\infty(\Gamma)$, which is a weak limit of states. A weak limit of states also determines a Loeb measure on the non-standard universe, ${}^*\Gamma$. Recall that ${}^*\Gamma$ is the space of all sequences in Γ , modulo eventual equality in the ultrafilter. A typical example of a Loeb measure is the uniform Loeb counting measure ([16]).

The Loeb counting measures are obtained as follows. Given a sequence of finite, positive integers (α_n) , this determines a Loeb measure μ_α . For any ultraproduct of finite subsets $(A_n)_{n \in \mathbb{N}}$ of Γ , the Loeb measure of the ultraproduct is obtained, by comparing, in the limit after the ultrafilter ω , their cardinality with the given sequence (α_n) .

Thus the Loeb measure of a hyperfinite set $C_\omega((A_n)_{n \in \mathbb{N}})$, by which we denote the ultraproduct of a family of finite sets $A_n \subseteq \Gamma, n \in \mathbb{N}$, is given by the formula:

$$(7) \quad \mu_\alpha(C_\omega(A_n)) = \lim_{n \rightarrow \omega} \frac{\text{card} A_n}{\alpha_n}.$$

The Loeb construction ([16]) proves, using the Caratheodory theorem and the finite stationarity of an increasing union of hyperfinite sets as above

(the \aleph_1 - saturation principle), that μ_α extends to a σ -algebra of measurable subsets of ${}^*\Gamma$, that are aproximable by hyperfinite sets ([16], [15]).

The passage to the Loeb measure formalism for representing states of $C^*((G \times G^{\text{op}}) \ltimes C(K))$ coming from its representation into the Calkin algebra, has the advantage that it allows to interpret quantities of the form

$$\lim_{n \rightarrow \omega} \frac{\text{card}(g_1 A_n g_2 \cap A_n)}{\text{card} A_n}, g_1, g_2 \in G.$$

which intervene in the calculation of the states, as the Koopmann representation matrix coefficients of a measure preserving action of $G \times G^{\text{op}}$ on a σ -finite Loeb measure space associated with the measures μ_α as above.

We prove that by taking convex combinations, and weak limits, we reduces the analysis of the states $\lim_{n \rightarrow \omega} \langle \cdot, \xi_n, \xi_n \rangle$ on $\pi_{\text{Koop}}(C^*((G \times G^{\text{op}}) \ltimes C(K)))$ to the analysis of the Koopmann representation of $C^*((G \times G^{\text{op}}) \ltimes l^\infty(\Gamma))$ acting on $L^2(\mathcal{Y}_A, \nu_\alpha)$, where the space \mathcal{Y}_A is the reunion of the translates by $G \times G^{\text{op}}$ of given hyperfinite set, and ν_α is te Koopmann measure.

So over the analysis original state is reduced to the analysis of states of the form

$$\langle \cdot, \chi_F, \chi_F \rangle_{L^2(\mathcal{Y}_\nu, A)}.$$

Note that the measure space \mathcal{Y}_ν is a measurable fibration over $\beta\Gamma$. If (A_n) avoids eventually any given finite set, then this is a fibration over $\partial(\beta\Gamma)$. F is a finite subset of \mathcal{Y}_A corresponding to hyperfinite set.

We analyse the Koopmann representation of $C^*((G \times G^{\text{op}}) \ltimes L^\infty(\mathcal{Y}_A, \nu_\alpha))$ on $L^2(\mathcal{Y}_A, \nu_\alpha)$ by disintegrating over the center \mathcal{Z} of the von Neumann algebra generated by $C^*((\Gamma \times 1) \ltimes L^\infty(\mathcal{Y}_A, \nu_\alpha))$.

By exactity of Γ , the von Neumann algebras in the disintegration are hyperfinite, and G induces a transformation group on \mathcal{Z} .

We prove that the transformation group (G, \mathcal{Z}) is itself a Loeb space, contained in ${}^*\Gamma$.

Here G acts simply by conjugation. The fact that conjugacy group are amenable completes the proof. More precisely, the dynamics of the action by conjugation of G on modular subgroups of Γ , forces the action of G to be amenable on \mathcal{Z} and hence on $C^*((\Gamma \times 1) \ltimes L^\infty(\mathcal{Y}_A, \nu_\alpha))$.

2. LOEB MEASURES AND STATES ON THE CROSSED PRODUCT ALGEBRA

In this section, we use the framework of Loeb measure to determine the structure of states on the Roe ([28]) C^* -algebra $C^*(\Gamma \ltimes l^\infty(\Gamma))$ and its extensions obtained by taking crossed products by larger groups.

Let X be a countable set and let \mathcal{G} be a countable discrete group. We assume that \mathcal{G} acts by partial permutations on X in the following way. For any $g \in \mathcal{G}$, we are given subsets $\mathcal{D}(g), \mathcal{R}(g)$ of X . We assume that g acts by determining a bijection, from $\mathcal{D}(g)$ onto $\mathcal{R}(g)$, denoted by $x \rightarrow g \cdot x$, for $x \in \mathcal{D}(g)$.

We assume that for all $g_1, g_2 \in \mathcal{G}$, the bijection induced by $g_1 g_2$ extends the (partial) composition of the bijections associated with g_1, g_2 . Thus, the composition domain

$$\{x \in \mathcal{D}(g_1) \mid g_1 x \in \mathcal{D}(g_2)\}$$

is contained in $\mathcal{D}(g_1 g_2)$ for all $g_1, g_2 \in \mathcal{G}$.

Then \mathcal{G} determines a canonical groupoid crossed product C^* -algebra $C^*(\mathcal{G} \ltimes l^\infty(X))$. This generalizes the usual notion of the Roe C^* -algebra.

We introduce the following definition that extends the usual definition of Koopmann representation.

Definition 2. Assume that \mathcal{G} acts as above, by (partial) measure preserving transformation on a (possibly infinite measure space) (\mathcal{Y}, ν) . Then each $g \in \mathcal{G}$ determines a partial isometry v_g acting on $L^2(\mathcal{Y}, \nu)$. We let $L^\infty(\mathcal{Y}, \nu)$ act by multiplication on $L^2(\mathcal{Y}, \nu)$. In this representation, the initial and the range space of the isometries $v_g, g \in \mathcal{G}$ are the characteristic functions of the domain and the range of the transformation induced by g on \mathcal{Y} .

Consider the groupoid crossed product C^* - algebra $C^*(\mathcal{G} \ltimes L^\infty(\mathcal{Y}, \nu))$. Then, the Koopmann representation

$$\pi_{\text{Koop}, \mathcal{Y}} : C^*(\mathcal{G} \ltimes L^\infty(\mathcal{Y}, \nu)) \rightarrow \mathcal{B}(L^2(\mathcal{Y}, \nu))$$

is, by definition, the representation of the algebra $C^*(\mathcal{G} \ltimes L^\infty(\mathcal{Y}, \nu))$, determined by representing \mathcal{G} into $\mathcal{B}(L^2(\mathcal{Y}, \nu))$, using the partial isometries $v_g, g \in \mathcal{G}$ above and letting $L^\infty(\mathcal{Y}, \nu)$ as above.

Example 3. If Γ is a discrete group, and $C^*(\Gamma \times l^\infty(\Gamma))$ is the Roe - algebra ([27]), then it is well known (see e.g. [5]) that the representation $\pi_{\text{Koop}, \Gamma}$ factorizes to an isomorphism from the reduced C^* - algebra $C_{\text{red}}^*(\Gamma \ltimes l^\infty(\Gamma))$ into $\mathcal{B}(l^2(\Gamma))$.

In the context introduced above, we endow the discrete set X with the counting measure. We consider the C^* - algebra $C^*(\mathcal{G} \ltimes X)$ and its Koopmann representation

$$\pi_{\text{Koop}, X} : C^*(\mathcal{G} \ltimes l^\infty(X)) \rightarrow \mathcal{B}(l^2(X)).$$

We want to determine the expression of the states on $C^*(\mathcal{G} \ltimes l^\infty(X))$ that are factorizing through the representation $\pi_{\text{Calk}} \circ \pi_{\text{Koop}}$.

These states are obtained by composing states on the Calkin algebra $\mathcal{Q}(l^2(X))$ with $\pi_{\text{Calk}} \circ \pi_{\text{Koop}}$. By [6], to obtain such states one considers an ultrafilter ω and a sequence $(\xi_n) \in l^2(X)$, that converges weakly to zero.

From the above, it is clear that:

Lemma 4. *Let ω be a free ultrafilter on \mathbb{N} . The C^* -states on the groupoid crossed product C^* -algebra $C^*(\mathcal{G} \rtimes X)$ factorizing through the representation $(\pi_{\text{Calk}} \circ \pi_{\text{Koop}})$ are of the form*

$$\varphi_{\omega, \xi}(a) = \lim_{n \rightarrow \omega} \langle \pi_{\text{Koop}}(a) \xi_n, \xi_n \rangle.$$

We may, for purpose of determining the norm induced by the representation $\pi_{\text{Calk}} \circ \pi_{\text{Koop}}$, assume that $\xi_n(x) \geq 0$ for all $x \in X$ and that the vectors $\xi_n, n \in \mathbb{N}$, viewed as functions on X have finite support.

When determining the C^* - norm on the image of $C^*(\mathcal{G} \rtimes l^\infty(X))$ through $\pi_{\text{Calk}} \circ \pi_{\text{Koop}}$, we will have to compute ultrafilter limits of matrix coefficients

$$(8) \quad \langle \pi_{\text{Koop}}(g) \xi_n, \xi_n \rangle, g \in \mathcal{G},$$

The restriction of the state $\varphi_{\omega, \xi}$ to $l^\infty(X)$ gives a measure ν_ξ on the Stone Cech compactification $\beta(X)$. More generally, given two sequences of unit vectors $\xi = (\xi_n)_{n \in \mathbb{N}}, \eta = (\eta_n)_{n \in \mathbb{N}}$, in $\ell^2(X)$, one defines a continuous functional on $C^*(\mathcal{G} \rtimes l^\infty(X))$, by the formula

$$(9) \quad \varphi_{\omega, \xi, \eta} = \lim_{n \rightarrow \omega} \langle \pi_{\text{Koop}}(g) \xi_n, \eta_n \rangle, g \in \mathcal{G}.$$

In particular this induces a continuous functional on $\ell^\infty(X)$ which corresponds to a finite, complex valued measure $\nu_{\xi, \eta}$ on $\beta(X)$, defined by the formula

$$(10) \quad \int_X f d \nu_{\xi, \eta} = \varphi_{\omega, \xi, \eta}(f), f \in C(\beta(X)).$$

For a partial transformation $\bar{g} = (g_1, g_2) \in \mathcal{G}$ we denote by \bar{g}^* the push-back operation on measures. To determine the coefficients from formula (8) we have to determine the behavior of translates of this measure with respect to

original measure. More precisely we have to determine the Radon Nykodim derivatives

$$\frac{d\bar{g}^*(\nu_\xi)}{d\nu_\xi}, \bar{g} \in \mathcal{G},$$

taken on its admissible domain.

To do this it is not sufficient to have only the information on the measure ν_ξ itself, but rather we have to keep track of the sequences determining the limit. This is a natural procedure in the context of the non-standard spaces.

We will prove, using Loeb measures, that the the GNS representations corresponding to states as in formula 8, although not necessary being Koopman, may be embedded by considering a larger space than $\beta(X)$ in a Koopman representation. Because of that we can do a reduction over the center of the larger algebra, obtained as the image of the larger Koopman representation.

Let *X be the space of sequences $\{(x_n) \mid x_n \in X, n \in \mathbb{N}\}$ factorized by the equivalence relation requiring that two sequences are equal if they coincide in the ultrafilter.

There is a natural projection $\pi_{\omega, X} : {}^*X \rightarrow \beta(X)$, associating to every sequence $(x_n) \in {}^*X$ the character $\varepsilon = \varepsilon(x_n)$ on $l^\infty(X)$ defined by the formula

$$\varepsilon(f) = \lim_{n \rightarrow \omega} f(x_n).$$

One considers a canonical family of subsets of *X . These are called hyperfinite sets. Given a family (A_n) of finite subsets of X , one defines

$$\mathcal{C}_\omega((A_n)_n) = \{(a_n) \mid a_n \in A_n \text{ } \omega\text{-eventually}\}.$$

On subsets of $\mathcal{C}_\omega((A_n)_n)$ one defines the uniform Loeb counting measure by defining for $\mathcal{C}_\omega(B_n) \subseteq \mathcal{C}_\omega(A_n)$

$$\mu_{\omega, \mathcal{C}_\omega(A_n)}(\mathcal{C}_\omega(B_n)) = \lim_{n \rightarrow \omega} \frac{\text{card} B_n}{\text{card} A_n}.$$

It is proven ([16], see also [15]) that $\mu_{\omega, \mathcal{C}_\omega(A_n)}$ extends to a measure to the σ - algebra of Loeb measure sets. Then any measurable set may be approximated with any degree of approximation by hyperfinite sets.

We will consider the σ - algebra ${}^*\alpha$ generated by all Loeb measurable sets, and consider the extension measure μ_α , which depends only on the cardinality $\alpha \in {}^*\mathbb{N}$ as follows.

Let $\alpha \in {}^*\mathbb{N}$ and define

$$\mu_\alpha(\mathcal{C}_\omega(A_n)) = \lim_{n \rightarrow \omega} \frac{\text{card} A_n}{\alpha_n}.$$

We introduce an equivalence relation on ${}^*\mathbb{N}$, by defining $\alpha \sim \beta$ if $\lim_{n \rightarrow \omega} \frac{\alpha_n}{\beta_n} \in (0, \infty)$ for $\alpha, \beta \in {}^*\mathbb{N}$.

Clearly, if $\alpha \approx \beta$ then the measure μ_α and μ_β are singular. We will consider a system of representatives \mathcal{N} for equivalence classes in ${}^*\mathbb{N}$.

The internal objects associated to the non-standard universe, are objects obtained through the ultralimit products. For example a sequence of functions $f_n : X \rightarrow X$ defines an interval function ${}^*f = (f_n) : {}^*X \rightarrow {}^*X$.

Also a sequence of probability measures μ_n on X of support A_n , defines an internal Loeb measure ${}^*\mu$ on $\mathcal{C}_\omega(A_n)$. The Loeb uniform counting measure is such an object.

We construct below a non-internal object. For a given $\alpha \in {}^*\mathbb{N}$, μ_α is a measure defined on all Loeb measurable sets in ${}^*\Gamma$. Assume that (ξ_n) is a sequence of vectors in $\ell^2(X)$ as in Lemma 1. Let $\lambda_n(x) = \xi_n^2(x)$, $x \in X$. Then λ_n is a sequence of probability measures on X . Let μ_α be the corresponding Loeb measure, whose support is the hyperfinite set $\mathcal{C}_\omega((\text{supp} \lambda_n)_n)$. We obtain

Lemma 5. *The measure μ_λ admits a direct decomposition with respect to the Loeb uniform counting measures. More precisely, there exists $F_\alpha \in$*

$L^1(*\Gamma, \mu_\alpha)$, whose support is a countable union of hyperfinite sets, for α in a countable subset \mathcal{N}_λ on $*\mathbb{N}$ such that

$$\mu_\lambda = \sum_{\alpha \in \mathcal{N}_\lambda} F_\alpha d\mu_\alpha.$$

The sum is well defined, since to different $\alpha \in *\mathcal{N}$, correspond mutually singular measures.

Proof. Choose a maximal family of mutually singular hyperfinite sets $\mathcal{C}_\omega(A_n^\beta)$ (that is the corresponding measures) such that $\mu_\lambda(\mathcal{C}_\omega(A_n^\beta)) \neq 0$ and $\mu_\lambda|_{\mathcal{C}_\omega(A_n^\beta)}$ is absolutely continuous with respect to Loeb counting measure on $\mathcal{C}_\omega(A_n^\beta)$.

If $\mu_\lambda(\mathcal{C}_\omega(A_n)) \setminus \bigcup \mathcal{C}_\omega(A_n^\beta)$ is non zero, then there exists $\mathcal{C}_\omega(A_n^0)$ contained in $\mathcal{C}_\omega(A_n) \setminus \bigcup \mathcal{C}_\omega(A_n^\beta)$ such that $\mu_\lambda(\mathcal{C}_\omega(A_n^0)) \neq 0$. By restricting the support we may assume that $\sup \mu_\lambda|_{\mathcal{C}_\omega(A_n^0)}$ is $\mathcal{C}_\omega(A_n^0)$.

Consider $(A_n^0)^M = \{a_n \in A_n^0 \mid \lambda^n(a_n) \leq \frac{M}{\text{card} A_n^0}\}$.

Then there exists $M > 0$ such that $\mu_\lambda(\mathcal{C}_\omega(A_n^{0,M})) \neq 0$ otherwise the support is strictly smaller than $\mathcal{C}_\omega(A_n^0)$. But then $\mu_\lambda|_{\mathcal{C}_\omega(A_n^{0,M})}$ is absolutely continuous to counting measure of $\mathcal{C}_\omega(A_n^0)$ and this contradicts maximality.

If we do the reunion of the $\mathcal{C}_\omega(A_n^\beta)$ corresponding to equivalent cardinality, the result follows. □

3. THE REPRESENTATION $\pi_{\text{Calk}} \circ \pi_{\text{Koop}}$ OF THE C^* - ALGEBRA

$$C^*(\mathcal{G} \rtimes \ell^\infty(X))$$

Recall that there exists a canonical, automatically \mathcal{G} - equivariant, Borel-measurable, projection $\pi : *X \rightarrow \beta(X)$ ([?]).

Definition 6. Let $\alpha \in \mathcal{N}$ be a hyperfinite integer, let $\mathcal{C}_\omega((A_n^\alpha))$ be a collection of disjoint hyperfinite sets and let $\mathcal{Y}_{A^\alpha} \subseteq *\Gamma$ be the reunion of the translates by elements in \mathcal{G} of $\mathcal{C}_\omega((A_n^\alpha)_{n \in \mathbb{N}})$.

Then clearly $(\mathcal{Y}_{A^\alpha}, \nu_\alpha)$ is a measure space, and ν_α is a \mathcal{G} - invariant measure.

Obviously we have a module structure on functions on \mathcal{Y}_{A^α} over $\ell^\infty(X) = C(\beta(X))$. This is because there is a canonical \mathcal{G} equivariant embedding Φ of $C(\beta(X))$ into functions on *X , which then restricts to a \mathcal{G} equivariant morphism, of commutative C^* -algebras:

$$(11) \quad \Phi_{A^\alpha} : \ell^\infty(X) \rightarrow L^\infty(\mathcal{Y}_{A^\alpha}, \nu_\alpha).$$

Here the image of a characteristic function of a subset A of X through Φ_{A^α} is the characteristic function consisting of all

$$\{(x_n)_{n \in \mathbb{N}} \in \mathcal{Y}_{A^\alpha} \mid x_n \in A, \omega - \text{eventually}\}.$$

Since Φ_{A^α} is \mathcal{G} - equivariant, we obtain a representation

$$(12) \quad \Phi_{A^\alpha} : C^*(\mathcal{G} \ltimes \ell^\infty(X)) \rightarrow C^*(\mathcal{G} \ltimes L^\infty(\mathcal{Y}_{A^\alpha}, \nu_\alpha)).$$

We analyze the following C^* - algebra representation:

$$(13) \quad \pi_{\text{Koop}, A^\alpha} = \pi_{\text{Koop}} \circ \Phi_{A^\alpha} : C^*(\mathcal{G} \ltimes \ell^\infty(X)) \rightarrow \mathcal{B}(L^2(\mathcal{Y}_{A^\alpha}, \nu_\alpha)).$$

We prove in the next theorem that the collection of representations as above determines the C^* -algebra crossed product norm, in the Calkin algebra embedding, on $C^*(\mathcal{G} \ltimes \ell^\infty(X))$.

Theorem 7. *The representation $\pi_{\text{Calk}} \circ \pi_{\text{Koop}}$ of $C^*(\mathcal{G} \ltimes \ell^\infty(X))$ is weakly contained in the direct sum of all representations $\pi_{\text{Koop}, A^\alpha}$, $\alpha \in \mathcal{N}$, as introduced above, in formula (13). For each $\alpha \in \mathcal{N}$, the sets $A^\alpha = \mathcal{C}_\omega((A_n^\alpha)_{n \in \mathbb{N}})$ run over all possible sequences of finite sets in X , that eventually avoid any given, finite subset of X , of cardinality*

$$\text{card} A^\alpha = (\text{card} A_n^\alpha)_{n \in \mathbb{N}} \in \mathcal{N},$$

equivalent to α .

Proof. To any sequence $\xi = (\xi_n)_{n \in \mathbb{N}}$ of vectors in $\ell^2(X)$, weakly convergent to 0, and to any free ultrafilter ω we associate a state on $C^*(\mathcal{G} \ltimes \ell^\infty(X))$, as defined in formula (8). As proved above in Lemma 4, because of [6], these states determine the C^* - norm on the image of crossed product C^* -algebra $C^*(\mathcal{G} \ltimes \ell^\infty(X))$ in its representation into the Calkin algebra $\mathcal{Q}(\ell^2(X))$.

Let $\nu_{\xi, \xi}$ be the measure on $\beta(X)$ introduced in formula (10). We have the decomposition

$$\nu_{\xi, \xi} = \oplus F_\alpha d\nu_\alpha,$$

proved in Lemma 5, where $F_\alpha = (F_\alpha^n)_{n \in \mathbb{N}}$ are internal functions on *X .

Then

$$\nu_{\xi, g\xi} = \oplus_\beta F_\beta^{1/2} [(F_\beta^{1/2}) \circ g] d\nu_\beta,$$

because, we may take

$$\xi_n = \sum_\alpha (F_\alpha^n)^{1/2} \left(\frac{\chi_{A_n^\alpha}}{(\alpha_n)^{1/2}} \right), \quad n \in \mathbb{N}.$$

Hence, since the measure ν_α are mutually singular it follows that

$$\lim_{n \rightarrow \infty} \langle \pi_{\text{Koop}}(g) \xi_n, \xi_n \rangle = \sum_\alpha \langle \pi_{\text{Koop}}(g) F_\alpha^{1/2}, F_\alpha^{1/2} \rangle_{L^2(\mathcal{Y}_\alpha, \nu_\alpha)}.$$

□

Corollary 8. *The C^* - norm on the crossed product $C^*(\mathcal{G} \ltimes \ell^\infty(X))$ in the representation $\pi_{\text{Calk}} \circ \pi_{\text{Koop}}$ is equal to the the supremum of the norm on $C^*(\mathcal{G} \ltimes \ell^\infty(X))$ coming from the representation $\pi_{\text{Koop}} \circ \Phi_{A^\alpha}$ of $C^*(\mathcal{G} \ltimes \ell^\infty(X))$ into $\mathcal{B}(L^2(\mathcal{Y}_{A^\alpha}, \nu_\alpha))$.*

4. THE ANALYSIS OF THE CENTER OF THE VON NEUMANN ALGEBRA

$$\{\pi_{\text{Koop}, A^\alpha}(C^*((\Gamma \times 1^{\text{op}}) \ltimes L^\infty(\mathcal{Y}_A, \nu_A)))\}''$$

In this section we let the set X be the discrete group Γ and consider the representation $\pi_{\text{Koop}} : C^*((\Gamma \times 1^{\text{op}}) \ltimes \ell^\infty(\Gamma)) \rightarrow \mathcal{B}(\ell^2(\Gamma))$ introduced in the previous section. Let $(\mathcal{Y}_{A_\alpha}, \nu_\alpha)$ be a measure space as in the previous section and consider the representation, introduced in formula (13)

$$\pi_{\text{Koop}, A^\alpha} = \pi_{\text{Koop}} \circ \Phi_{A^\alpha},$$

restricted to $C^*((\Gamma \times 1) \ltimes \ell^\infty(\Gamma))$. We analyze the von Neumann subalgebra corresponding to the image, through the representation $\pi_{\text{Koop}, A^\alpha}$ of crossed product C^* -algebra $C^*((\Gamma \times 1^{\text{op}}) \ltimes \ell^\infty(\Gamma))$.

In this representation the groupoid \mathcal{G} decomposes as $((\Gamma \times 1) \ltimes \ell^\infty(\Gamma)) \ltimes G$ and will induce an action on the center that will be analysed in this section.

Lemma 9. *The crossed product C^* -algebra $C^*((\Gamma \times 1^{\text{op}}) \ltimes L^\infty(\mathcal{Y}_A, \nu_\alpha))$ is nuclear.*

Proof. Since Γ is exact, it follows that $C^*(\Gamma \ltimes \ell^\infty(\Gamma))$ is nuclear. But $L^\infty(\mathcal{Y}_A, \nu_A)$ is a $\ell^\infty(\Gamma)$ - C^* -algebra, and being nuclear, the result follows from [3]. \square

We analyze the center \mathcal{Z} of the von Neumann algebra

$$\{\pi_{\text{Koop}, A_\alpha}(C^*(\Gamma \times 1^{\text{op}}) \ltimes L^\infty(\mathcal{Y}_A, \nu_\alpha))\}''.$$

Because of the next section we may assume that the action of Γ on \mathcal{Y}_A is free, up to a finite group. Hence the center \mathcal{Z} is contained in $L^\infty(\mathcal{Y}_A, \nu_A)$ and consists of $\Gamma \times 1^{\text{op}}$ invariant functions.

To do this we will analyze the action of $\Gamma \times 1^{\text{op}}$ on $(\mathcal{Y}_A, \nu_\alpha)$. For this reason we consider the reduced crossed product on von Neumann algebra, which is of type I or II. We know, because of nuclearity that this algebra is hyperfinite and the center consists of Γ -invariant functions.

The following statement is true for an arbitrary exact i.c.c. discrete group Γ , with the following additional assumptions:

Definition 10 (Property \mathcal{A}). We say that the inclusion $\Gamma \subseteq G$ has property \mathcal{A} if the following two assumptions hold true:

- (i) The set of conjugacy \mathcal{C} classes of amenable subgroups of Γ is at most countable.
- (ii) For any infinite amenable subgroup Γ_0 of Γ , if $(\tilde{F}_n)_{n \in \mathbb{N}}$ is a relative sequence of Folner sets in G for Γ_0 , then there exist a sequence $(x_n)_{n \in \mathbb{N}}$ in G such that \tilde{F}_n is of the form $F_n x_n$ when $F_n \subseteq \Gamma_0$ are Folner sets for Γ_0 and $x_n \in \Gamma$, $n \in \mathbb{N}$.

In the next statement we are using in an essential way the fact that the reduced crossed product von Neumann algebra

$$\mathcal{L}((\Gamma \times 1^{\text{op}}) \ltimes L^\infty(\mathcal{Y}_A, \nu_\alpha))$$

is hyperfinite. We denote $\mathcal{Y} = \mathcal{Y}_A$, $\nu = \nu_\alpha$. We prove:

Theorem 11. *In the context introduced above, let Γ be an exact group for which the conditions in Definition 10 hold true. Assume that Γ acts freely on \mathcal{Y}_A . Let \mathcal{Z} be the center algebra of the von Neumann algebra $\mathcal{L}((\Gamma \times 1^{\text{op}}) \ltimes L^\infty(\mathcal{Y}, \nu))$. Then, there exists a central decomposition of the crossed product von Neumann algebra*

$$\mathcal{Z} = \mathcal{Z}_I \oplus \mathcal{Z}_{II}, \quad \mathcal{Z}_{II} = \bigoplus_{\Gamma_0 \in \mathcal{C}} \mathcal{Z}_{\Gamma_0}$$

corresponding to a Γ -invariant partition of the set \mathcal{Y} into

$$\mathcal{Y} = \mathcal{Y}_I \cup \mathcal{Y}_{II}, \quad \mathcal{Y}_{II} = \bigcup_{\Gamma_0 \in \mathcal{C}} \mathcal{Y}_{\Gamma_0},$$

with the following properties:

- *Type I.* Γ on \mathcal{Y}_I admits a fundamental domain F .
- *Type II.* For every $[\Gamma_0] \in \mathcal{C}$, \mathcal{Y}_{Γ_0} admits a Γ/Γ_0 wandering domain F_{Γ_0} .

If the action of Γ is free up to a finite group $\Gamma_1 \subseteq \Gamma$ then all of the above is still valid, if taken modulo Γ_1 .

Before proving the theorem we note that in the type II case, because of the Mackey induced representation construction ([17]), then, modulo a choice of coset representatives, we have a Γ -equivariant identification

$$L^\infty(\mathcal{Y}_{\Gamma_0}, \nu) = l^2(\Gamma/\Gamma_0) \otimes L^\infty(F_{\Gamma_0}, \nu).$$

Proof of Theorem II. We identify Γ with $\Gamma \times 1^{\text{op}}$. In the case of type I this is obvious, because, in this case, a projection in $L^\infty(\mathcal{Y}_I, \nu)$, that is minimal in $\mathcal{L}(\Gamma \ltimes \mathcal{Y}_I)$ will correspond to the characteristic function of a fundamental domain for the action of Γ on \mathcal{Y}_I .

Let $\mathcal{M} = \mathcal{L}(\Gamma \ltimes L^\infty(\mathcal{Y}_{II}))$ be the type II component. Since $C_{\text{red}}^*(\Gamma \ltimes L^\infty(\mathcal{Y}_{II}))$ is nuclear it follows that \mathcal{M} is hyperfinite ([8]). We disintegrate over the center \mathcal{Z} and analyze the structure of the corresponding II_∞ factor \mathcal{M}_z , for z a generic point in the spectrum of \mathcal{Z} . All the statements below hold true for z almost everywhere in the spectrum of \mathcal{Z} , with respect to the measure introduced in the next section.

Let D_z be the fiber at z corresponding to the disintegration of $L^\infty(\mathcal{Y}_A, \nu)$. This is a Cartan subalgebra. We have a representation π_z of Γ into the unitary group of \mathcal{M}_z , which normalizes D_z .

Because \mathcal{M}_z is hyperfinite, using the unicity of the Cartan subalgebra ([9], see the proof in [25]), it follows that we can choose a family of countable sets X_z , such that $\mathcal{B}(\ell^2(X_z)) \cong \mathcal{B}(H_z)$ and such that D_z decomposes as $D_z^1 \otimes D_z^2$ and \mathcal{M}_z decomposes as $\mathcal{B}(H_z) \otimes \mathcal{N}_z$, where D_z^1 is the diagonal

algebra $\ell^\infty(X_z)$ in $\mathcal{B}(\ell^2(X_z))$ and D_z^2 is a Cartan subalgebra in the type II_1 factor \mathcal{N}_z . Clearly \mathcal{N}_z is hyperfinite.

The fiber at z of $L^\infty(\mathcal{Y}_A, \nu)$ which is isomorphic to $D_z^1 \times D_z^2$ carries a σ -finite measure ν_z , obtained by the disintegration of the measure ν , which is the tensor product of the counting measure on X_z and the restriction $(\nu_0)_z$ of the trace on \mathcal{N}_z to D_z^2 .

The group $\pi_z(\Gamma)$ will act as a permutation group on the diagonal algebra $\ell^\infty(X_z)$, tensor product a representation of Γ into \mathcal{N}_z . Clearly, for z -almost everywhere, there exists a subgroup $(\Gamma_0)_z$ of Γ such that the action on the first component X_z is conjugated to the action by left multiplication of Γ on $\Gamma/(\Gamma_0)_z$.

Then $(\Gamma_0)_z$ fixes the projection in $\ell^\infty(X_z)$ corresponding to the trivial coset. This corresponds to a decomposition of the action of $\pi_z(\Gamma)$ such that $D_z^1 \cong \ell^\infty(\Gamma/(\Gamma_0)_z)$, and $D_z^2 = L^\infty(F_z, (\nu_0)_z)$, with $(\Gamma_0)_z$ acting ergodically on $L^\infty(F_z, (\nu_0)_z)$ and such that the trace on \mathcal{M} disintegrates as the tensor product of the canonical traces on $\mathcal{B}(\ell^2(X_z))$ and of the canonical trace on

$$\mathcal{N}_z \cong \mathcal{L}((\Gamma_0)_z \ltimes L^\infty(F_z, (\nu_0)_z)).$$

Using the Assumption (ii) in Definition 10, we obtain a partition of \mathcal{Z} as in the statement, such that each element in the partition corresponding to a class of a group Γ_0 is the characteristic function of the measurable sets consisting of all z in the spectrum of \mathcal{Z} such that $(\Gamma_0)_z$ is conjugated in Γ to Γ_0 .

To this decomposition corresponds a partition of \mathcal{Y}_A as in the statement. This uniquely determines sets $F_{\Gamma_0} \subseteq \mathcal{Y}_{\Gamma_0}$, whose fiber at z is the set F_z , for z as above.

The set $F_{\Gamma_0} \subseteq \mathcal{Y}_{\Gamma_0}$ may be replaced by a reunion of hyperfinite sets. The Cantor diagonal to the sets in the reunion of hyperfinite sets, being eventually invariant by Γ_0 , yields a family of Følner sets. Hence, because of assumption

(ii) in Property \mathcal{A} , we may assume that $F_{\Gamma_0} \subseteq \mathcal{C}_\omega((\Gamma'_0 x_n))$ for some choice of $x_n \in \Gamma$, $n \in \mathbb{N}$, where Γ'_0 is a maximal abelian subgroup in Γ as in Section 5. The proof is now completed by using Lemma 14 and the argument in the proof of Corollary 15 in Section 5. □

5. THE NON FREE PART OF THE ACTION OF $(G \times G^{\text{op}})$ ON ${}^*\Gamma$

If C is a subset of Γ , we denote by C' is the centralizer subgroup of C in Γ :

$$C' = \{\gamma \in \Gamma \mid \gamma c \gamma^{-1} = c, \text{ for all } c \in C\}.$$

If C is more generally a subset of G we extend this notion by letting C' be the set of all $g \in G$ having the same property as above. It is obvious to see that in the case we are considering in this paper, that is when Γ is the modular group, it follows that all the stabilizer subgroups are always abelian and cyclic.

Lemma 12. *Let $G \times G^{\text{op}}$ act on Γ by left and right multiplication. Fix $(g_1, g_2) \in G \times G^{\text{op}}$. Assume that (g_1, g_2) keeps x fixed, that is $g_1 x g_2^{-1} = x$ (equivalently $g_2 = x^{-1} g_1 x$).*

Then, the set of points fixed by (g_1, g_2) is the coset $\{g_1\}'x = x\{g_2\}'$.

Proof. Let y be another point fixed by (g_1, g_2) . Then

$$g_2 = x^{-1} g_1 x = y^{-1} g_1 y.$$

Hence yx^{-1} commutes to

$$\Gamma_1 = \{g_1\}'.$$

Consequently y belongs to $\{g_1\}'x = x\{g_2\}'$. The reciprocal is along the same line of argument. □

The previous result has the following obvious corollary:

Corollary 13. *Assume $a = (a_n)_{n \in \mathbb{N}} \in \mathcal{Y}_A$ is fixed by an element $(g_1, g_2) \in G \times G^{\text{op}}$. Then there exist a coset C of the subgroup $\{g_1\}' \cap \Gamma$ such that*

$$a_n \in C, \omega\text{--eventually for } n \in \mathbb{N}.$$

Lemma 14. *Let H be a discrete group. and let Γ_0, Γ_1 be two subgroups. Let x_0, x_1 be two elements H . which determine the cosets $\Gamma_0 x_0, \Gamma_1 x_1$, Assume that the intersection $\Gamma_0 x_0 \cap \Gamma_1 x_1$ is non-void. Let x be a point in the above intersection.*

Then

$$\Gamma_0 x_1 \cap \Gamma_1 x_1 = (\Gamma_0 \cap \Gamma_1)x.$$

Proof. The set in the right hand side of the equality in the statement is obviously contained in the set of the left hand side of the equality. We prove the converse.

Let $x' \neq x$ be an element in the intersection $\Gamma_0 x_0 \cap \Gamma_1 x_1$, which, by hypothesis, contains x .

Consequently there exist $\gamma_0, \gamma'_0 \in \Gamma_0$ (respectively $\gamma_1, \gamma'_1 \in \Gamma_1$) such that

$$x = \gamma_0 x_0 = \gamma_1 x_1,$$

$$x' = \gamma'_0 x_0 = \gamma'_1 x_1.$$

Then

$$x(x')^{-1} = (\gamma_0 x_0)(\gamma'_0 x_0)^{-1} = \gamma_0(\gamma'_0)^{-1} \in \Gamma_0,$$

and

$$x(x')^{-1} = (\gamma_1 x_1)(\gamma'_1 x_1)^{-1} = \gamma_1(\gamma'_1)^{-1} \in \Gamma_1.$$

Hence

$$\gamma_0(\gamma'_0)^{-1} = \gamma_1(\gamma'_1)^{-1}.$$

We denote the common value from the above formula by θ . Then, by construction, we have that

$$\theta \in \Gamma_0 \cap \Gamma_1.$$

Since

$$x(x')^{-1} = \theta,$$

it follows that $x' = \theta^{-1}x$. Consequently x' belongs to $(\Gamma_0 \cap \Gamma_1)x$. \square

Because of the previous lemma, if Γ_0, Γ_1 are subgroups of Γ with trivial intersection, then $\Gamma_0 x \cap \Gamma_1 y$ consists of at most one point for all $x, y \in \Gamma$.

Corollary 15. *Let Γ_0 be an infinite, cyclic abelian subgroup of Γ . Let*

$$C = x\Gamma_0, x \in \Gamma$$

be a coset of Γ_0 . Let $\tilde{\Gamma}_0$ be the unique maximal abelian subgroup of G containing Γ_0 . Assume $(g_1, g_2) \in G \times G^{\text{op}}$ has the property that

$$g_1 C g_2 \cap C$$

has cardinality ≥ 2 . Then

$$g_1 \in \tilde{\Gamma}_0, g_2 \in x^{-1}\tilde{\Gamma}_0 x.$$

Proof. This is a direct consequence of the previous corollary and of the fact that maximal abelian, infinite subgroups of Γ are either disjoint, modulo the identity element or either are equal. \square

Lemma 16. *Let Γ_0 be a maximal abelian subgroup of Γ . Assume Γ_0 is infinite. Let $\tilde{\Gamma}_0$ be the unique maximal abelian subgroup of G containing Γ_0 . Let \mathcal{C}_{Γ_0} be the family consisting of all cosets of subgroups of Γ that are conjugated, by groups elements in G , to Γ_0 .*

Let P_{Γ_0} be the minimal projection in the universal W^* algebra associated to $\pi_{\text{Calk}}(C^*((G \times G^{\text{op}}) \ltimes \ell^\infty(\Gamma)))$ that dominates all the projections $\chi_C, C \in \mathcal{C}_{\Gamma_0}$. Then

- (i) P_{Γ_0} is a central projection in the universal W^* -algebra associated to $\pi_{\text{Calk}}(C^*((G \times G^{\text{op}}) \ltimes \ell^\infty(\Gamma)))$.
- (ii) $P_{\Gamma_0} \pi_{\text{Calk}}$ factorizes to a representation of $C_{\text{red}}^*((G \times G^{\text{op}}) \ltimes \ell^\infty(\Gamma))$.

Proof. The fact that P_{Γ_0} is a central projection follows from the fact that any element $(g_1, g_2) \in G \times G^{\text{op}}$ will map a coset of Γ_0 inside another coset of a subgroup conjugated to Γ_0 .

Because of the preceding corollary the partial action of $G \times G^{\text{op}}$ on Γ , is isomorphic to a partial action of $G \times G^{\text{op}}$ on $\Gamma/\Gamma_0 \times \Gamma/\Gamma_0 \times \Gamma_0$, via the Mackey induction construction. Since Γ_0 is amenable the result follows. \square

Corollary 17. Let \mathcal{Y}_A^0 be the part of \mathcal{Y}_A consisting of points having non trivial, infinite stabilizer under the partial action of $G \times G^{\text{op}}$. Then

- (i) The Hilbert space $L^2(\mathcal{Y}_A^0, \nu_\alpha)$ is left invariant by the representation π_{A^α} of $C^*((G \times G^{\text{op}}) \ltimes \ell^\infty(\Gamma))$ introduced in formula (13).
- (ii). Denote by $\pi_{A^\alpha}^0$ the induced representation of $C^*((G \times G^{\text{op}}) \ltimes \ell^\infty(\Gamma))$ into $\mathcal{B}(L^2(\mathcal{Y}_A^0, \nu_\alpha))$. Then the representation $\pi_{A^\alpha}^0$ factorizes to a representation of $C_{\text{red}}^*((G \times G^{\text{op}}) \ltimes \ell^\infty(\Gamma))$.

Proof. Because Γ is a free product of groups, any commutant $\{\gamma\}' \cap \Gamma$ is an abelian cyclic group. Moreover this is maximal abelian in Γ . By Corollary 13, \mathcal{Y}_A^0 is contained in the image of P_{Γ_0} for some $\Gamma_0 \in \mathcal{C}_{\Gamma_0}$. The result follows then from the previous statement. \square

6. A PLANCHEREL TYPE MEASURE ON THE CENTER \mathcal{Z} OF THE VON NEUMANN ALGEBRA $\mathcal{L}(\Gamma \ltimes L^\infty(\mathcal{Y}_A, \nu_A))$

In this section we construct a, Plancherel like, measure $\tilde{\nu}$ on \mathcal{Z} . We will prove in the next section that this is invariant to the action of G on \mathcal{Z} .

Definition 18 (The Plancherel measure in type I case). Consider a Γ -invariant, measurable subset \tilde{F} of the spectrum of \mathcal{Z} . Its characteristic function defines a projection in \mathcal{Z} . Then there exists a subset F of \tilde{F} with the following two properties:

- (i) The central support projection of χ_F in $\mathcal{L}(\Gamma \ltimes L^\infty(\mathcal{Y}_A, \nu_A))$ is $\chi_{\tilde{F}}$.
- (ii) For every $\gamma \in \Gamma \setminus \{1\}$ we have $\nu(F \cap \gamma F) = 0$. We will refer to this property of the set F with the terminology Γ -wandering.

We define the Plancherel measure $\tilde{\nu}$ of \tilde{F} to be the $\nu_\alpha(F)$.

Definition 19 (The Plancherel measure in the type II case). Let \tilde{F} be a Γ -invariant subset \tilde{F} of the spectrum of \mathcal{Z}_{II} corresponding to an amenable subgroup $\Gamma_0 \in \mathcal{C}$. The characteristic function $\chi_{\tilde{F}}$ defines a projection in \mathcal{Z}_{Γ_0} . Then there exists a subset F_{Γ_0} of \tilde{F} with the following two properties:

- (i) The central support projection of $\chi_{F_{\Gamma_0}}$ in $\mathcal{L}(\Gamma \ltimes L^\infty(\mathcal{Y}_A, \nu_A))$ is $\chi_{\tilde{F}}$.
- (ii) For every $\gamma \in \Gamma \setminus \Gamma_0$ we have $\nu(F_{\Gamma_0} \cap \gamma F_{\Gamma_0}) = 0$. We will refer to this property of the set F_{Γ_0} with the terminology Γ/Γ_0 -wandering.
- (iii) The set F_{Γ_0} is left invariant by Γ_0 .

We let in this case $\tilde{\nu}(\tilde{F})$ be $\nu_\alpha(F_{\Gamma_0})$.

Lemma 20. *The measure in the type II case is independent of the choice of Γ_0 , as long as we impose that we choose a maximal group Γ_0 with the above properties.*

Proof. Indeed the cosets spaces $[\Gamma_0 x]$ and $[\Gamma_1 y]$, if Γ_0, Γ_1 are maximal amenable, (and hence cyclic) have finite intersection unless $\Gamma_0 = \Gamma_1$. Thus if Γ admits a Γ_0 - invariant, Γ/Γ_0 fundamental domain, then \mathcal{Y}_A will be covered with translates of sets $\mathcal{C}_\omega((\gamma\Gamma_0\gamma^{-1}x_n))$, $\gamma \in \Gamma$, which are disjoint to the similar sets associated to a group Γ_1 , not conjugated with Γ_0 . On the other hand, if $\Gamma_1 = x\Gamma_0x^{-1}$, the fundamental domains for Γ_1 are transformed into xF .

Hence the measure is well defined.

If $\Gamma_0 \subseteq \Gamma_1$ both cyclic, and $[\Gamma_0 x_n]$ are Følner sets for Γ_0 , then these are Følner sets for Γ'_0 too. Hence we may always work with Γ'_0 . Thus the Plancherel measure is uniquely determined if we use the maximal Γ_0 .

□

In the rest of the section we will use the isomorphism:

$$(14) \quad C^*((G \times G^{\text{op}}) \ltimes L^\infty(\mathcal{Y}, \nu)) = C^*(C^*(\Gamma \ltimes L^\infty(\mathcal{Y}, \nu)) \rtimes \Delta(G)),$$

where $\Delta(G) = \{(g, g^{-1}) \mid g \in G\}$, and $\Delta(G)$ acts by partial isomorphisms on $C^*(\Gamma \ltimes L^\infty(\mathcal{Y}, \nu))$, as explained in the next definition. Here Γ is identified with $\Gamma \times 1^{\text{op}} \subseteq G \times G^{\text{op}}$.

Recall that K is the profinite completion of Γ with respect to the normal subgroups determined by Γ_g , $g \in G$. For $\sigma \in G$, let $K_\sigma \subseteq K$ be the closure in the profinite topology of Γ_σ . In the specific case of G and Γ considered in this paper we have

$$K_\sigma = K \cap \sigma K \sigma^{-1}.$$

Definition 21. For σ in G , the action of $(\sigma, \sigma^{-1}) \in G \times G^{\text{op}}$ on \mathcal{Y} , induces a trace preserving, partial action denoted by $\sigma \cdot \sigma^{-1}$, on $\mathcal{L}(\Gamma \ltimes L^\infty(\mathcal{Y}, \nu))$, as described below. This in turn induces a partial action on the center \mathcal{Z} .

Let π be the projection from \mathcal{Y} onto $\beta\Gamma$, and compose it with the canonical projection from $\beta\Gamma$ onto K . We denote this projection by $\pi_{\mathcal{Y}}$. The Γ -equivariance of the map $\pi_{\mathcal{Y}}$ implies that left convolution by elements in $\Gamma_{\sigma^{-1}}$ leaves $\pi_{\mathcal{Y}}^{-1}(K_{\sigma^{-1}})$ invariant, and similarly for Γ_{σ} .

The domain of the partial action (σ, σ^{-1}) is $\mathcal{L}(\Gamma_{\sigma^{-1}} \ltimes \pi_{\mathcal{Y}}^{-1}(K_{\sigma^{-1}}))$. Then $\sigma \cdot \sigma^{-1}$ will map $\mathcal{L}(\Gamma_{\sigma^{-1}} \ltimes \pi_{\mathcal{Y}}^{-1}(K_{\sigma^{-1}}))$ onto $\mathcal{L}(\Gamma_{\sigma} \ltimes \pi_{\mathcal{Y}}^{-1}(K_{\sigma}))$.

The, the center algebras $\mathcal{Z}(\Gamma_{\sigma^{-1}})$, $\mathcal{Z}(\Gamma_{\sigma})$ of the von Neumann algebras $\mathcal{L}(\Gamma_{\sigma^{-1}} \ltimes \pi_{\mathcal{Y}}^{-1}(K_{\sigma^{-1}}))$, and respectively $\mathcal{L}(\Gamma_{\sigma} \ltimes \pi_{\mathcal{Y}}^{-1}(K_{\sigma}))$ consist of $\Gamma_{\sigma^{-1}}$ invariant, measurable, functions on $\pi_{\mathcal{Y}}^{-1}(K_{\sigma^{-1}})$ and respectively Γ_{σ} invariant, measurable functions in $\pi_{\mathcal{Y}}^{-1}(K_{\sigma})$. Clearly (σ, σ^{-1}) maps $\mathcal{Z}(\Gamma_{\sigma^{-1}})$ onto $\mathcal{Z}(\Gamma_{\sigma})$.

We prove bellow that the action introduced in the preceding definition extends to a representation of $G \cong \Delta(G)$ into the trace preserving transformations of \mathcal{Z} .

Proposition 22. *The partial transformations $\sigma \cdot \sigma^{-1}$, $\sigma \in G$, introduced in Definition 21 extend to a representation of $G \cong \Delta(G)$, for which we will use by extension also the notation $\sigma \cdot \sigma^{-1}$, into the trace preserving transformations of the center algebra \mathcal{Z} of the von Neumann algebra $\mathcal{L}((\Gamma \times 1^{\text{op}}) \ltimes L^{\infty}(\mathcal{Y}_A, \nu_{\alpha}))$.*

Proof. A projection \mathcal{Z} is represented as the characteristic function $\chi_{\tilde{F}}$ of a Γ -invariant measurable set \tilde{F} .

In the case of type I, there exists a Γ - wandering subset F of \mathcal{Y}_I such that $\tilde{F} = \Gamma F$. Let t_i be a selection of the coset representatives for Γ_{σ} in Γ . Then \tilde{F} may be alternatively obtained as ΓF_0 , by taking

$$F_0 = \bigcup t_i^{-1}(\pi_{\mathcal{Y}}^{-1}(t_i K_{\sigma^{-1}}) \cap F) = \bigcup (\pi_{\mathcal{Y}}^{-1}(K_{\sigma^{-1}}) \cap t_i^{-1} F).$$

Consequently in the case of type I, the center \mathcal{Z} is identified with the center of the von Neumann algebra $\mathcal{L}(\Gamma_\sigma \ltimes L^\infty(\mathcal{Y}_I, \nu))$ and the action $\sigma \cdot \sigma^{-1}$ of $\sigma \in G$, will map $\tilde{F} = \Gamma F$ into the Γ_σ invariant set

$$F_1 = \bigcup_{\gamma \in \Gamma_\sigma} \sigma F_0 \sigma^{-1}.$$

Let r_j be the coset representatives for Γ_σ in Γ . This is extended to the Γ -invariant set $\bigcup_{r_j} \bigcup_{\gamma \in \Gamma_\sigma} r_j(\sigma F_0 \sigma^{-1})$. The action of $\sigma \cdot \sigma^{-1}$ is consequently defined by letting, for \tilde{F}, F, F_0 as above

$$(15) \quad (\sigma \cdot \sigma^{-1})(\Gamma F) = \bigcup_{r_j} \bigcup_{\gamma \in \Gamma_\sigma} r_j(\sigma F_0 \sigma^{-1}).$$

Since F was chosen to be a Γ -wandering subset of \mathcal{Y} , the right hand side in formula (15) depends only on \tilde{F} and this correspondence defines a representation of G by transformations of \mathcal{Z} .

Next we analyze the type II case. Consider amenable subgroup $\Gamma_0 \in \mathcal{C}$. Assume that $\chi_{\tilde{F}}$ is an idempotent in \mathcal{Z}_{Γ_0} , where $\tilde{F} = \bigcup_{\gamma \in \Gamma/\Gamma_0} \gamma F$, and F is a Γ/Γ_0 wandering, measurable subset of \mathcal{Y} .

We identify $\chi_{\tilde{F}}$ with an element in the center $\mathcal{Z}(\Gamma_{\sigma^{-1}})$ of the von Neumann algebra $\mathcal{L}(\Gamma_{\sigma^{-1}} \ltimes \pi_{\mathcal{Y}}^{-1}(K_{\sigma^{-1}}))$. To do this we choose a system t_i of coset representatives for $\Gamma \subseteq \Gamma_{\sigma^{-1}}$.

Then

$$\tilde{F} = (\Gamma/\Gamma_0)F.$$

In the above equality we may replace F by the measurable subset F_1 given by the formula

$$F_1 = \bigcup t_i(\pi_{\mathcal{Y}}^{-1}(t_i^{-1}K_{\sigma^{-1}}) \cap F) = \bigcup \pi_{\mathcal{Y}}^{-1}(K_{\sigma^{-1}}) \cap t_i F.$$

The sets $t_i F$ are left invariant by left convolution by elements in the group

$$(16) \quad \Gamma_i = \Gamma_{\sigma^{-1}} \cap (t_i \Gamma_0 t_i^{-1}).$$

By the second fundamental group isomorphism theorem we have that

$$\frac{\Gamma_0}{\Gamma_0 \cap \Gamma_\sigma} \cong \frac{\Gamma_\sigma \Gamma_0}{\Gamma_\sigma}.$$

Hence

$$[\Gamma_0 : \Gamma_0 \cap \Gamma_\sigma] = [\Gamma : \Gamma_\sigma].$$

Consequently $\Gamma_0 \cap \Gamma_\sigma$ is an infinite abelian group. The same holds true for the groups $\Gamma_i = \Gamma_\sigma \cap t_i \Gamma_0 t_i^{-1}$. Let

$$F_i = \pi_y^{-1}(K_{\sigma^{-1}}) \cap t_i F.$$

Then the sets F_i form a partition of F_1 and they are fixed by the groups Γ_i , introduced in formula (16). For all i these sets are also Γ/Γ_i wandering. This is because F_i is contained in $\pi_y^{-1}(K_{\sigma^{-1}})$ and any element in $\Gamma \setminus \Gamma_\sigma$ will move it into a set disjoint to F_1 . Let

$$\tilde{F}_i = \bigcup_{\gamma \in \Gamma/\Gamma_i} \gamma F_i.$$

Then the characteristic function of \tilde{F}_i is a central projection in the type II component of $\mathcal{Z}(\Gamma_{\sigma^{-1}})$, corresponding to the Γ_i component to the type II part of the center.

To the central projection $\chi_{\tilde{F}}$ we are now associating a central projection in $\mathcal{Z}(\Gamma_{\sigma^{-1}})$, which is the characteristic function of the $\Gamma_{\sigma^{-1}}$ -invariant set

$$\tilde{F}_1 = \cup \tilde{F}_i.$$

The new expression for \tilde{F} is:

$$(17) \quad \tilde{F} = \cup t_i \tilde{F}_i.$$

The action of $\sigma \cdot \sigma^{-1}$ on the Γ -invariant set \tilde{F} will give a set

$$\tilde{F}_\sigma = \sigma \tilde{F} \sigma^{-1},$$

that is Γ_σ -invariant, which is obtained as reunion over all left translations by elements in Γ_σ of the set

$$\bigcup_i \sigma(\pi_y^{-1}(K_{\sigma^{-1}}) \cap t_i F) \sigma^{-1} = \bigcup_i \pi_y^{-1}(K_\sigma) \cap \sigma t_i F \sigma^{-1}.$$

Here the sets $\sigma t_i F \sigma^{-1}$ are left invariant by $\sigma \Gamma_i \sigma^{-1}$ and are $\Gamma / \sigma \Gamma_i \sigma^{-1}$ wandering. Hence \widetilde{F}_σ is a Γ_σ -invariant subset whose characteristic functions defines a projection in the type II component of $\mathcal{Z}(\Gamma_\sigma)$ corresponding to the amenable subgroup $\sigma \Gamma_i \sigma^{-1}$.

To obtain a Γ -equivariant set, which is our definition for $\sigma \widetilde{F} \sigma^{-1}$ one proceeds as in formula (17), by using this time Γ_σ coset representatives. As in the type I case this gives a representation of G into the group of transformations of the type II component of \mathcal{Z} .

The above transformation group is trace preserving since in each of the two cases the measure on \mathcal{Z} is determined by comparison with the measure of the corresponding fundamental domains in the type I case and respectively with the measure of Γ / Γ_0 wandering domain in the case of type II. Indeed, both quantities are preserved by the partial transformations on \mathcal{Y} induced by $G \times G^{\text{op}}$. The transformations $\sigma \cdot \sigma^{-1}$ on cZ , $\sigma \in G$ are obtained by putting together portions of the transformations on \mathcal{Y} and hence they preserve the trace. \square

7. REALIZATION OF CENTER \mathcal{Z} OF THE VON NEUMANN ALGEBRA

$\mathcal{L}((\Gamma \times 1^{\text{op}}) \ltimes L^\infty(\mathcal{Y}_A, \nu_\alpha))$ AS THE FIBER OVER THE NEUTRAL
ELEMENT IN THE PROFINITE COMPLETION OF Γ

In this section, using the fact that the crossed product algebra

$$\mathcal{L}((\Gamma \times 1^{\text{op}}) \ltimes L^\infty(\mathcal{Y}_A, \nu_\alpha))$$

contains the crossed product $\mathcal{L}((\Gamma \times 1^{\text{op}}) \ltimes C(K))$, we prove that the center

$$\mathcal{Z} = \mathcal{Z}(\mathcal{L}((\Gamma \times 1^{\text{op}}) \ltimes L^\infty(\mathcal{Y}_A, \nu_\alpha))),$$

is isomorphic to the fiber at e of the measurable fibered space

$$\mathcal{Y}_A \xrightarrow{\pi_{\mathcal{Y}_A}} K.$$

From an intuitive point of view, this is natural, since every Γ - invariant function, is constant along the fibration over K .

We will also determine the dynamics of the G - measure space \mathcal{Z} in terms of data regarding wandering subsets. Here the action of G on \mathcal{Z} is the diagonal type action introduced in the last section. As in the previous section, we denote this action in by $g \cdot g^{-1}$, $g \in G$.

Let $\nu_P = \tilde{\nu}$, the Plancherel measure on \mathcal{Z} , introduced in Definitions 18, 19. Let $\tilde{F}_0, \tilde{F}_1, \dots, \tilde{F}_n$ be Γ -invariant subsets of \mathcal{Y} . To determine the action of G on \mathcal{Z} we compute moments of the form

$$(18) \quad \nu_P(\tilde{F}_0 \cap g_1 \tilde{F}_1 g_1^{-1} \dots \cap g_n \tilde{F}_n g_n^{-1}), g_1, g_2, \dots, g_n \in G.$$

We prove that the expression above is as limit of a sum of similar moments, for subsets of \mathcal{Y} with respect to the $G \times G^{\text{op}}$ -invariant measure ν on \mathcal{Y} .

For $n \in \mathbb{N}$, let a_i^n be a system of coset representatives, for the subgroup $\Gamma_{\sigma_{p^n}}$ in Γ . Let $\Gamma_n, n \in \mathbb{N}$ be the finite index normal subgroup of Γ given by the formula

$$\Gamma_n = \bigcap_i a_i^n \Gamma_{\sigma_{p^n}} (a_i^n)^{-1}.$$

It is well known ([19]) that the family $(\Gamma_n)_{n \in \mathbb{N}}$ defines the profinite completion K of Γ . For $\sigma \in G$, we choose a system (t_i^σ) of coset representatives for

Γ_σ in Γ . For each $n \in \mathbb{N}$, we choose a system of representatives, depending on the cases of type I or II, introduced in Theorem 11.

In the case of type I, we let (s_i^n) be a system of coset representatives for Γ_n in Γ . Let $\Gamma_0 \in \mathcal{C}$ be an amenable subgroup of Γ determining a type II component \mathcal{Z}_{Γ_0} of the center \mathcal{Z} as introduced in Theorem 11. Since $\Gamma_n, n \in \mathbb{N}$ is a normal subgroup of Γ it follows that $\Gamma_n \Gamma_0 = \Gamma_0 \Gamma_n, n \in \mathbb{N}$ are subgroups of Γ . In addition the family $(\Gamma_n \Gamma_0)_{n \in \mathbb{N}}$ is decreasing.

In this case, the projective limit of the increasing family of quotients $(\Gamma/(\Gamma_n \Gamma_0))$,

$$K_{\Gamma_0} = \lim_{n \rightarrow \infty} \Gamma/(\Gamma_n \Gamma_0),$$

becomes a totally disconnected, compact space.

Let $s \in \Gamma$ and let Γ_n be a group in the family $(\Gamma_n)_{n \in \mathbb{N}}$. In the case of type I, we denote by $\overline{s\Gamma_n}$ the image of the coset $s\Gamma_n$ in the projective limit defining the profinite completion K of Γ . In the case of type II, we denote by $\overline{s\Gamma_n}$ the image of the coset $s\Gamma_n \Gamma_0$ in the projective limit. We have chosen this notation in order to have a uniform notation for both the case of type I or of type II.

Similarly to the projection $\pi_{\mathcal{Y}} : \mathcal{Y} \rightarrow K$, introduced in Definition 21 we may construct in this case a $G \times G^{\text{op}}$ -equivariant, measurable projection $\pi_{\mathcal{Y}, \Gamma_0} : \mathcal{Y} \rightarrow K_{\Gamma_0}$.

In the case of type II we let (s_i^n) be a system of coset representatives for the subgroup $\Gamma_n \Gamma_0$ in Γ .

Consider the Γ -invariant subsets $\tilde{F}_0, \tilde{F}_1, \dots, \tilde{F}_k \subseteq \mathcal{Y}$, whose characteristic functions determine central projections in \mathcal{Z} . Because of Theorem 11 there exists measurable subsets F_0, F_1, \dots, F_k of \mathcal{Y} such that $\tilde{F}_i = \Gamma F_i$ and

such that the sets F_i are Γ -wandering in the case of type I and respectively, Γ/Γ_0 - wandering and left invariant by Γ_0 in the type II case.

Using the choices introduced above we have

Proposition 23. *The measure $\nu_P(\tilde{F}_0 \cap \tilde{F}_1 \cap \dots \cap \tilde{F}_k)$ is computed, in the case of type I, as the limit, when $n \rightarrow \infty$,*

$$\sum_{i_0, \dots, i_k} \nu(s_{i_0}^n [\pi_Y^{-1}((\overline{s_{i_0}^n})^{-1} \Gamma_n \cap F_0)] \cap \dots \cap s_{i_k}^n [\pi_Y^{-1}((\overline{s_{i_k}^n})^{-1} \Gamma_n \cap F_k)]).$$

In the case of type II, for the component of the center \mathcal{Z} corresponding to a subgroup $\Gamma_0 \in \mathcal{C}$, one has to replace, in the above formula, the projection π_Y by the projection π_{Y, Γ_0} .

Proof. This is simply a consequence of the fact that for every n in \mathbb{N} , we are replacing the representative for the set F_0, F_1, \dots, F_k generating $\tilde{F}_0, \tilde{F}_1, \dots, \tilde{F}_k$ by an equivalent one contained in $\pi_Y^{-1}(\overline{\Gamma}_n)$ and respectively, in the type II case, contained in $\pi_{Y, \Gamma_0}^{-1}(\overline{\Gamma}_n)$.

□

We remark that one could use, instead of a single family of subgroups $(\Gamma_n)_{n \in \mathbb{N}}$, for each $l = 0, 1, \dots, k$, a different family $(\Gamma_n^l)_{n \in \mathbb{N}}$ of finite index, normal subgroups, and consequently different systems of coset representatives for each l .

The above proposition proves that indeed the computation of moments with respect to the measure ν_P , as in formula (18), is reduced to the computation of generalized moments that are computed with respect to the initial measure ν . The sets intervening in the computation of the moments do concentrate, in the limit, to the fiber of \mathcal{Y} standing over the identity element of the profinite completion K and respectively image of the identity in K_{Γ_0} in the type II case.

Lemma 24. *We use the notations and definitions introduced above. Consider the generalized moment*

$$\nu_P(\tilde{F}_0 \cap g_1 \tilde{F}_1 g_1^{-1} \cap \dots \cap g_n \tilde{F}_n g_n^{-1}), \quad g_1, g_2, \dots, g_n \in G,$$

where the notation $g_i \cdot g_1^{-1}, \dots, g_n \cdot g_n^{-1}$ refers to the action, on the center \mathcal{Z} , introduced in Lemma 22.

Then the above expression is equal to

$$(19) \quad \sum_{i_0, \dots, i_k} \nu(s_{i_0}^n [\pi_Y^{-1}((\overline{s_{i_0}^n})^{-1} \Gamma_n \cap F_0)] \cap \dots \cap g_k [s_{i_k}^n [\pi_Y^{-1}((\overline{s_{i_k}^n})^{-1} \Gamma_n \cap F_k)]] g_k^{-1}).$$

In the second later the action of G is derived from the action of $G \times G^{\text{op}}$ on \mathcal{Y} .

As in the previous statement, in the case of type II, for the component of the center \mathcal{Z} corresponding to a subgroup $\Gamma_0 \in \mathcal{C}$, one has to replace, in the above formula, the projection π_Y by the projection π_{Y, Γ_0} introduced above.

Proof. By definition, the measure ν_P of any intersection of Γ -invariant sets $\tilde{F}_0, \tilde{F}_1, \dots, \tilde{F}_n$, is determined by the measure of the Γ -wandering (respectively Γ/Γ_0 wandering) sets needed to generate the intersection. The procedure in Proposition 32 is to construct a sequence of equivalent (in the sense considered in [20]), generating sets. There is no limit of the sequence of sets, but applying the measure ν one obtains a constant measure. Each set in the reunion represents a contribution to the center, (eventually, taken for a smaller fixing amenable subgroup in the type II case)

The effect of the action of $g \cdot g^{-1}$ on such a sequence, as in formula (19) transforms the sequence of reunion of generating sets into a reunion of smaller subsets, concentrating in the limit, into the fiber at e , that are still generating, by taking the reunion after left translations over Γ the same set. By the construction of the action of G on \mathcal{Z} this reunion is $\tilde{F}_0 \cap g_1 \tilde{F}_1 g_1^{-1}, \dots, g_k \tilde{F}_k g_k^{-1}$,

and using again Proposition 32 and the remark noted after its proof, one obtains the equality in the statement. \square

An alternative way of proving the previous result would be to consider the space $K \times_{\Gamma} \mathcal{Y}$, where Γ acts by right multiplication on K . This space has a natural $G \times G^{\text{op}}$ -equivariant, measurable fibration over K . The fiber at e may be identified with the sequence of generating sets, and writing the action of (g, g^{-1}) , which preserves the fiber at the identity element e would give the proof of the previous statement.

8. THE CENTER OF THE ALGEBRA $\mathcal{L}((\Gamma \times 1) \ltimes L^{\infty}(\mathcal{Y}_A, \nu_{\alpha}))$ IS
EQUVARIANTLY ISOMORPHIC WITH A LOEB MEASURE SPACE

In this section we use the computations in the last section for the generalized moments of the measure ν_P on the center, to identify the G -measure space (\mathcal{Z}, ν_P) with a Loeb G -measure space. We prove the following.

Proposition 25. *Let \mathcal{Z} be, as in the preceding sections, the center of the von Neumann algebra $\mathcal{L}((\Gamma \times \{1\}^{\text{op}}) \ltimes (\mathcal{Y}_A, \nu_{\alpha}))$, which as proved in the previous section is also the center of the von Neumann algebra*

$$\{\pi_{\text{Koop}}(C^*((\Gamma \times \{1\}^{\text{op}}) \ltimes L^{\infty}(\mathcal{Y}_A, \nu_{\alpha})))\}''.$$

Let \tilde{F} be a Γ -invariant, measurable subset of \mathcal{Y}_A , of finite Plancherel measure $\tilde{\nu}$.

Then, for every $\varepsilon > 0$ and for every natural number N there exists a family of normal subgroups Γ_n , in the family of finite index normal subgroups defining the profinite completion K , and there exists hyperfinite sets

$$\mathcal{C}_{\omega}((\tilde{A}_n)_{n \in \mathbb{N}}) \subseteq \mathcal{C}_{\omega}((\Gamma_n)_{n \in \mathbb{N}}),$$

such that any first N in an enumeration of the sequence of the generalized moments of \tilde{F} with respect to ν_P and G , as in introduced in Lemma 24, are approximated up to order ε , with the corresponding generalized moments of $\mathcal{C}_\omega(\tilde{A}_n)$ with respect to G and the uniform Loeb measure ν_α .

Proof. In the case of type I, $\tilde{F} = \Gamma F$, where $F \subseteq \mathcal{Y}$ is a Γ -wandering subset of \mathcal{Y}_A . By approximating, we may assume F is a hyperfinite set. In the case of type II, we consider families $\mathcal{C}_\omega((A_n^s)_{n \in \mathbb{N}})$ approximating F , and hence, for every $\varepsilon > 0$, we get a family $(A_n)_{n \in \mathbb{N}}$ such that Γ_0 , almost invariants, up to order ε and N , the set $\mathcal{C}_\omega((A_n)_{n \in \mathbb{N}})$ and such that $\mathcal{C}_\omega((A_n)_{n \in \mathbb{N}})$ is almost Γ/Γ_0 -wandering.

Choose an enumeration g_1, g_2, \dots, g_n of G and choose a sequence $\varepsilon_n \searrow 0$. For each Γ_n , let s_i^n be a choice of coset representatives for Γ_n in Γ . Then there exists a sufficiently large k_n such that

$$\text{card}(A_{k_n} \cap s_i^n \Gamma_n) / \text{card} A_n$$

and the corresponding measure formulae, in translates by g_1, g_2, \dots, g_n in intersections up to order N , are equal, up to ε_n , to the measures of the corresponding intersections involving F and the measure ν up to order n .

Let

$$(20) \quad \tilde{A}_n = \bigcup_i (s_i^n)^{-1} ((s_i^n \Gamma_n) \cap A_{k_n}), n \in \mathbb{N}.$$

Then, because of the formula in the preceding section, we have that for all $g_1, g_2, \dots, g_n \in G$, the quantity

$$\nu_P(\tilde{F} \cap g_1 \tilde{F} g_1^{-1} \cap \dots \cap g_n \tilde{F} g_n^{-1})$$

is equal, up to order ε_n , to

$$\mu_\alpha(\mathcal{C}_\omega(\tilde{A}_n) \cap g_1 \mathcal{C}_\omega(\tilde{A}_n) g_1^{-1} \cap \dots \cap g_n \mathcal{C}_\omega(\tilde{A}_n) g_n^{-1}).$$

Corollary 26. *Consequently, as a G -measure space, the center algebra \mathcal{Z} , endowed with the measure ν_P is G -measurably isomorphic to a G -Loeb measure space $(\mathcal{Y}_{(\tilde{A}_n)_{n \in \mathbb{N}}}, \nu_{\text{card} \tilde{A}})$ having the property that there exists a family of normal, finite index subgroups (Γ_n) , with trivial intersection, such that*

$$\tilde{A}_n \subseteq \Gamma_n, \quad \omega - \text{ for } n \text{ eventually.}$$

In particular, we may assume

$$\mathcal{Y}_{\tilde{A}} \subseteq \bigcup_{g \in G} (g \cdot g^{-1}) \mathcal{C}_{\omega}(\tilde{A}).$$

□

9. THE REPRESENTATION OF THE ALGEBRA $C^*((G \times G^{\text{op}} \ltimes C(K)))$ INTO THE CALKIN ALGEBRA $Q(\ell^2(\Gamma))$ FACTORIZES TO

$$C_{\text{red}}^*((G \times G^{\text{op}}) \ltimes C(K))$$

In this section we prove that the representation $\pi_{\text{Calk}} \circ \pi_{\text{Koop}}$ of the crossed product C^* -algebra $C^*((G \times G^{\text{op}}) \ltimes C(K))$ factorizes to the reduced crossed product C^* -algebra.

A possible argument for this could be obtained on the following lines:

The groupoid C^* -crossed product $C^*((G \times G^{\text{op}}) \rtimes C(K))$ is isomorphic to the groupoid crossed product C^* -algebra $C^*(\Delta(G) \ltimes C^*((\Gamma \times 1^{\text{op}}) \ltimes C(K)))$ where $\Delta(g), g \in G$ acts on $C^*((\Gamma \times 1^{\text{op}}) \ltimes C(K))$ as described in formula (3), by partial isomorphisms.

A similar formula holds in the Koopmann representation. Indeed, by the same arguments, the C^* -algebra

$$(21) \quad \mathcal{A}_{A,\alpha} = C^*((G \times G^{\text{op}}) \times L^\infty(\mathcal{Y}_A, \nu_\alpha))$$

is isomorphic to

$$C^*(\Delta(G) \ltimes C^*((\Gamma \times 1^{\text{op}}) \ltimes L^\infty(\mathcal{Y}_A, \nu_\alpha))).$$

But, in this representation, the action of $\Delta(G)$ on the von Neumann algebra center

$$(22) \quad \mathcal{Z}_A = \mathcal{Z}(\{\pi_{\text{Koop}, \mathcal{Y}_A}(C^*(\Gamma \times 1) \ltimes L^\infty(\mathcal{Y}_A, \nu_\alpha))\}'')$$

is approximated by the action of $\Delta(G)$ on $L^\infty(\mathcal{Y}_{\tilde{A}}, \nu_\alpha)$, where the finite sets family $\tilde{A} = (\tilde{A}_n)_{n \in \mathbb{N}}$ was introduced in the previous section.

Because $\mathcal{Y}_{\tilde{A}}$ is a measurable fibration over $\beta(G)$ and G acts (by conjugating) with amenable stabilizers on the conjugation orbits, it follows that $\Delta(G)$ acts amenably on the center, in the sense that the crossed product algebra is nuclear.

Following this line of arguments it would follow that the C^* -algebra $\mathcal{A}_{A, \alpha}$ introduced in formula (21) is nuclear and hence its image through the Koopman representation is also nuclear. This forces that the corresponding representation of $C^*(\Delta(G) \ltimes C^*(\Gamma \times 1^{\text{op}}) \times C(K))$ to factorize to the C_{red}^* representation.

Instead of using this line of reasoning, we consider the following approach:

Theorem 27. *The action $\Delta(G)$ on the center algebra \mathcal{Z}_A introduced in Proposition 22 has the property that for all measurable subsets \tilde{F} of \mathcal{Z}_A , of finite measure ν_P , there exists sequence of measurable subsets $\tilde{F}_n \subseteq \tilde{F}$, $n \in \mathbb{N}$, such that, for each n , the moments*

$$g \rightarrow \nu_\alpha(\Delta(g)\tilde{F}_n \cap \tilde{F}_n), g \in G$$

have support contained in a finite reunion of double cosets of Γ in G .

The proof of this theorem will be divided in several steps in this section. We first note that this will end the proof of Theorem 1.

Proof of Theorem 1. Let \mathcal{Y}_A be constructed as in Definition 6. Assume that the set \tilde{F} , is obtained by the procedure described in Theorem 11 from a finite measure subset $F \subseteq \mathcal{Y}_A$, that is Γ -wandering in the case of type I, respectively Γ/Γ_0 -wandering, for some amenable subgroup $\Gamma_0 \in \mathcal{C}$ in the case of type II. Let $(\tilde{F}_n)_{n \in \mathbb{N}}$ be as in the statement of Theorem 27 and let $F_n = F \cap \tilde{F}_n$. In this case the state determined on $C^*((G \times G^{\text{op}}) \rtimes C(K))$, in the Koopman representation $\pi_{\text{Koop}, A}$, by the vector $\frac{1}{\nu_\alpha(F_n)}^{1/2} \chi_F \in L^2(\mathcal{Y}_A, \nu_\alpha)$ will have support in a finite reunion of pairs of double cosets of Γ contained in $G \times G^{\text{op}}$

Since, by the \mathcal{S} property of Ozawa ([23]) we know that $C^*((\Gamma \times \Gamma^{\text{op}}) \rtimes C(K))$, in the Koopman representation factorizes to a representation of the $C_{\text{red}}^*((\Gamma \times \Gamma^{\text{op}}) \rtimes C(K))$, it follows that the representation $\pi_{\text{Calk}} \circ \pi_{\text{Koop}}$ of $C^*((G \times G^{\text{op}}) \rtimes C(K))$ is weakly contained in the C_{red}^* representation. \square

Recall that $\Gamma_{p^s}^0$ is the maximal (finite index) normal subgroup contained in

$$\Gamma_{p^s} = \Gamma \cap \sigma_{p^s} \Gamma \sigma_{p^s}^{-1}.$$

To prove Theorem 27 we let $\tilde{F} = \mathcal{C}_\omega((\tilde{A}_n)_{n \in \mathbb{N}})$ be a hyperfinite set as considered in the previous section. Because of the result in the previous section we may assume that we are given a strictly increasing sequence $(s_n)_{n \in \mathbb{N}}$ such that

$$(23) \quad \tilde{A}_n \subseteq \Gamma_n = \Gamma_{p^{s_n}}^0, n \in \mathbb{N}.$$

Because of the condition in formula (23), and since the action by conjugation of σ_{p^n} on G will map $\Gamma_{p^s}^0 \subseteq \Gamma$ back into Γ , for $s > n$, $s, n \in \mathbb{N}$, it follows that G , induces, via the conjugation action Δ , a transformation group on $\mathcal{Y}_{\tilde{A}}$. We analyze the conjugation action of $\Delta(G)$ on conjugation orbits of Γ , that are contained in the space $\mathcal{Y}_{\tilde{A}}$.

For x in Γ denote by O_x^Γ (respectively O_x^G) the orbit of x , under the conjugation action, by Γ (respectively by G).

It is obvious that for $g \in \Gamma\sigma\Gamma$, $gO_xg^{-1} \cap \Gamma_n$ is non void and only if O_x intersects $x \in \Gamma_n \cap g^{-1}\Gamma_ng$, i.e. if, O_x intersects $\Gamma_n \cap \sigma^{-1}\Gamma_n\sigma$. For $e \in \{1, 2, \dots\}$ we consider, for $n \in \mathbb{N}$,

$$\tilde{A}_n^e = \{a \in \tilde{A}_n \mid O_a^\Gamma \text{ does not intersect } \Gamma_n \cap (\sigma_{p^{e+1}})^{-1}\Gamma_n\sigma_{p^{e+1}}\}.$$

Lemma 28. *We let \tilde{F}^e be the the hyperfinite subset of $\mathcal{C}_\omega(\tilde{A})$ defined by the formula*

$$(24) \quad \tilde{F}^e = \mathcal{C}_\omega((\tilde{A}_n^e)_n), e \in \mathbb{N}$$

Let \tilde{F}_∞ be the measurable subset defined by the formula

$$\tilde{F}_\infty = \mathcal{C}_\omega(\tilde{A}) \setminus \left[\bigcup_e \tilde{F}^e \right].$$

Thus \tilde{F}_∞ is the subset of $\mathcal{C}_\omega(\tilde{A})$ consisting of all sequences $(a_n)_n$ in $\mathcal{C}_\omega(\tilde{A})$ such that for every integer k , the set

$$\{n \mid O_{a_n}^\Gamma \text{ intersects } \Gamma_n \cap (\sigma_{p^k})^{-1}\Gamma_n\sigma_{p^k}\}$$

is cofinal in ω .

Then

- (1). *The sets $\bigcup_e \tilde{F}^e$ and \tilde{F}_∞ have disjoint G orbits in $\mathcal{Y}_{\tilde{A}}$.*
- (2). *The intersection $(\Gamma\sigma_{pf}\Gamma)\tilde{F}^e \cap \tilde{F}^e$ is non-void only if $f \leq e$.*
- (3). *The states*

$$g \rightarrow \frac{1}{\mu(\tilde{F}_e)} \langle g(\tilde{F}^e), \tilde{F}^e \rangle$$

are $C_{\text{red}}^(G)$ continuous and verify the conditions of Theorem 27.*

Proof. We denote the coset representatives for $\Gamma_{\sigma_{p^e}}$ in Γ by $s_i^e, i = 1, 2, \dots, n_e$. Clearly

$$\mathcal{C}_\omega(\tilde{A}_n^e) = \mathcal{C}_\omega(\tilde{A}_n) \cap [\mathcal{C}_\omega(\Gamma_n) \setminus \bigcup_{i=1}^{n_e} \mathcal{C}_\omega((s_i^e \sigma_{p^e} \Gamma_n (\sigma_{p^e})^{-1} (s_i^e)^{-1}))].$$

The hyperfinite set $\mathcal{C}_\omega(\tilde{A}_n^e)$ has obviously the property that its translates by elements in $(\Gamma_{\sigma_{p^f}} \Gamma)$, for $f > e$, has void intersection with $\mathcal{C}_\omega(\tilde{A}_n^e)$ and this proves property (2) in the statement

□

It remains to analyze the state corresponding to \tilde{F}_∞ . Using the above choices we have:

$$\tilde{F}_\infty = \mathcal{C}_\omega(\tilde{A}_n) \cap \bigcap_{e \geq 1} \mathcal{C}_\omega(\Gamma_n \cap (\bigcup_{i=1}^{n_e} (s_i^e \sigma_{p^e} \Gamma_n (\sigma_{p^e})^{-1} (s_i^e)^{-1}))).$$

To prove Theorem 27 it remains to prove that the state on G corresponding to displacement of \tilde{F}_∞ is continuous with respect to the $C_{\text{red}}^*(G)$ norm and verifies the conditions of Theorem 27, we introduce the following definition

We will prove bellow that the dynamics of the action of G on the remaining part of $\mathcal{Y}_{\tilde{A}}$ is governed by the action, by conjugation, of G on the space of subgroups of the form Γ_g of Γ .

Let \mathbf{Z}_p are the p -adic integers. Let $K = PSL_2(\mathbf{Z}_p)$ be the maximal compact subgroup of $PGL_2(\mathbb{Q}_p)$. We introduce the following structure.

Definition 29. Let G act by conjugation on $PGL_2(\mathbb{Q}_p)$. Let \mathcal{MS} be the minimal, G -equivariant, σ - algebra of subsets of $PGL_2(\mathbb{Q}_p)$ containing K . By $\mathcal{MS} \cap K$, we denote the Borel algebra obtained by intersecting all the sets in \mathcal{MS} with K .

Clearly \mathcal{MS} contains all intersections $K_g = K \cap gKg^{-1}$, $g \in G$. Moreover G acts on \mathcal{MS} and $\mathcal{MS} \cap K$ by conjugation and $\mathcal{MS} \cap K$ is left invariant by Γ .

We first prove a "nesting" property for the subgroups, whose characteristic function generate \mathcal{MS} .

Lemma 30. *For g in G , the subgroup K_g is uniquely determined by the coset $s\sigma_{p^e}\Gamma$ to which g belongs. Moreover, there exists an order preserving equivalence between such subgroups and the set of cosets of $\Gamma\sigma_{p^e}$, $e \geq 1$ in Γ , ordered by inclusion:*

If g belongs to $s\sigma_{p^e}\Gamma$, and $s\Gamma_{\sigma_{p^e}}$ is contained in $s_1\Gamma_{\sigma_{p^{e-1}}}$ then for any g_1 in $s_1\sigma_{p^{e-1}}\Gamma$ we have $K_g \subseteq K_{g_1}$.

Proof. This is equivalent to the corresponding property of the subgroups $\Gamma_g = g\Gamma g^{-1} \cap \Gamma$ of Γ and this property is almost tautological. Indeed

$$s\Gamma_{\sigma_{p^e}}s^{-1} = \Gamma_{s\sigma_{p^e}}, s \in \Gamma, e \geq 1.$$

On the other hand for g in G , $\gamma \in \Gamma$ we have $\Gamma_{g\gamma} = \Gamma_g$.

If s belongs to $\Gamma_{\sigma_{p^{e-1}}}$ then $s\Gamma_{\sigma_{p^{e-1}}}s^{-1} = \Gamma_{\sigma_{p^{e-1}}}$ and hence

$$\Gamma_{s\sigma_{p^e}} = s\Gamma_{\sigma_{p^e}}s^{-1} \subseteq s\Gamma_{\sigma_{p^{e-1}}} = \Gamma_{\sigma_{p^{e-1}}}.$$

□

Because of the "nesting" property, it follows that any infinite intersection of sets in \mathcal{MS} , reintersected with K , will contain a reunion of infinite intersections of the form $K \cap K_{s_1\sigma_{p^1}} \cap \dots \cap K_{s_e\sigma_{p^e}} \cap \dots$ where $s_e\Gamma_{\sigma_{p^e}} \subseteq s_{e-1}\Gamma_{\sigma_{p^{e-1}}}$ for all $e \geq 1$.

Lemma 31. *There is a one to one correspondence*

$$(y, t) \rightarrow K_{(y,t)}.$$

between infinite intersection as above with points (y, t) in $P^1(\mathbf{Z}_p^2)$.

Proof. Such a decreasing sequence of cosets corresponds to a coset in K/K_∞ , where K_∞ consists of the lower triangular matrices in K , that is the subgroup of matrices of the form $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$ in K .

Hence the intersection is determined uniquely by an element in the projective space $P^1(\mathbf{Z}_p^2)$.

Indeed if $s_e = \begin{pmatrix} x_e & y_e \\ z_e & t_e \end{pmatrix}$, modulo the scalars, then $s_e \Gamma_{\sigma_{p^e}}$ is determined by $(y_e, t_e) \in P^1(\mathbb{Z}_{p^e}^2)$, and the nesting condition

$$s_e \Gamma_{\sigma_{p^e}} \subseteq s_{e-1} \Gamma_{\sigma_{p^{e-1}}}$$

corresponds to the fact that for $e \geq 1$, $(y_e, z_e) \equiv (y_{e-1}, z_{e-1})$ in $P^1(\mathbb{Z}_{p^{e-1}}^2)$. \square

We analyze now the structure of infinite intersections.

Lemma 32. *Given two distinct points (y_1, t_1) and (y_2, t_2) in $P^1(\mathbf{Z}_p^2)$, the intersection $K_{(y_1, t_1)} \cap K_{(y_2, t_2)}$ will reintersect a third subset of the form $K_{(y_3, t_3)}$, with (y_3, t_3) in $P^1(\mathbf{Z}_p^2) \setminus \cup_{i=1,2} (y_i, t_i)$, in the trivial element.*

Proof. By left translations by elements in K , we may assume that $(y_1, t_1) = (0, 1)$ in $P^1(\mathbf{Z}_p^2)$ and thus $K_{0,1} = K \cap \bigcap_{e \geq 1} K_{\sigma_{p^e}} = K_\infty$.

Assume that $\begin{pmatrix} x_2 & y_2 \\ z_2 & t_2 \end{pmatrix}$ is a representative in $K = \text{PSL}_2(2, \mathbf{Z}_p)$ of the coset of K/K_∞ represented by $(y_2, t_2) \in P^1(\mathbf{Z}_p^2)$.

Then K_{y_2, t_2} is $\begin{pmatrix} x_2 & y_2 \\ z_2 & t_2 \end{pmatrix} K_\infty \begin{pmatrix} x_2 & y_2 \\ z_2 & t_2 \end{pmatrix}^{-1}$ and hence $K_{(0,1)} \cap K_{(y_2, t_2)}$ consists of all elements $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$ in $K_{(0,1)}$ such that

$$\begin{pmatrix} x_1 & y_1 \\ z_1 & t_1 \end{pmatrix}^{-1} \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ z_1 & t_1 \end{pmatrix}$$

.

This condition becomes in \mathbf{Z}_p

$$y_1 t_1 (a - d) = y_1^2 c.$$

Thus, if $(0, 1) \neq (y_1, t_1)$ in $P^1(\mathbf{Z}_p^2)$, the intersection $K_{(0,1)} \cap K_{(y, t)}$ is :

$$\left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(2, \mathbf{Z}_p) \mid t_1(a - d) = y_1 c \right\}.$$

Clearly this can reintersect $K_{(0,1)} \cap K_{(y_2, t_2)}$ in a non-trivial element if and only if $(y_2, t_2) = (y_1, t_1)$ in $P^1(\mathbf{Z}_p^2)$. \square

In the following we describe a \mathcal{MS} module structure on the measure space $(\mathcal{Y}_{\tilde{A}}, \nu_\alpha)$. Recall that the group Γ_{p^n} is the kernel of the surjection $\mathrm{PSL}(2, \mathbb{Z}) \rightarrow \mathrm{PSL}(2, \mathbb{Z}_{p^n})$.

Definition 33. For a family of subgroups H_n of Γ let $\mathcal{C}_\omega((H_n))$ consist of all sequences $(\gamma_n)_n$, such that γ_n belongs to H_n eventually, with respect to the ultrafilter ω .

Let s_n be a strictly increasing sequence of natural numbers and let $\Gamma_n = \Gamma(p^{s_n})$. Then (Γ_n) is a decreasing sequence of normal subgroups of Γ , with trivial intersection.

Let $\mathcal{MS}_\omega((\Gamma_n))$, which, for simplicity, when no confusion is possible, we denote by \mathcal{MS}_ω , be the minimal, G -invariant (with respect the adjoint

action) σ -algebra of subsets of ${}^*\Gamma$, containing $\mathcal{C}_\omega((\Gamma_n)_n)$. Because of the G -invariance requirement this automatically contains their conjugates

$$\mathcal{C}_\omega((g\Gamma_n g^{-1})_n) = g\mathcal{C}_\omega((\Gamma_n)_n)g^{-1}, g \in G.$$

Note that although the adjoint action of G does not map Γ into itself, the sets $\mathcal{C}_\omega((\Gamma_n)_n)$ are eventually mapped back into ${}^*\Gamma$.

By $\mathcal{MS}_\omega \cap \mathcal{C}_\omega((\Gamma_n)_n)$ we denote the subsets of \mathcal{MS}_ω that are contained in $\mathcal{C}_\omega((\Gamma_n)_n)$.

Note that G acts, by the adjoint operation, by partial transformations, both on $\mathcal{MS}_\omega \cap \mathcal{C}_\omega((\Gamma_n)_n)$ and on $\mathcal{MS} \cap K$. In the next lemma we prove that modulo sequences of elements in Γ that converge very fast to the identity the two actions of G correspond to each other.

Lemma 34. *Then there exists a unique sets σ -algebra homeomorphism Φ_ω from \mathcal{MS}_ω onto \mathcal{MS} , subject to the the following requirements.*

(a). *The morphism Φ_ω is G - equivariant and maps the characteristic function of $\mathcal{C}_\omega((\Gamma_n)_n)$ into the characteristic function of $K = \text{PSL}(2, \mathbf{Z}_p)$.*

(b). *Φ_ω maps*

$$(25) \quad \bigcap_{e \geq 1} \mathcal{C}_\omega((\Gamma(p^{s_n+e}))_n).$$

into the identity element e of K .

Proof. Since every intersection of finite index subgroups is again a finite index subgroup, it follows that if $(H_n^s)_n$, $s \in \mathbb{N}$ is an infinite collection of decreasing sequences of finite index subgroups of Γ , then $\bigcap_s \mathcal{C}_\omega((H_n^s)_n)$ is always non trivial, as it contains

$$\mathcal{C}_\omega((H_n^1 \cap H_n^2 \cap \dots \cap H_n^n)_n).$$

Hence the only problem in establishing the homeomorphism from \mathcal{MS}_ω onto \mathcal{MS} will consist in determining the kernel of this correspondence.

To do this observe that the nesting property proven for the subsets $K_g, g \in G$ also holds true for the groups

$$A_g = \mathcal{C}_\omega((\Gamma_n \cap g\Gamma_n g^{-1})_n) = \mathcal{C}_\omega((\Gamma_n)) \cap g\mathcal{C}_\omega((\Gamma_n)_n)g^{-1}.$$

Indeed it is obvious that if g belongs to $s\sigma_{p^e}\Gamma$, then A_g depends only on $s\sigma_{p^e}$. Indeed $A_{g\gamma} = A_g$ for all $g \in G, \gamma \in \Gamma$ since computing $A_{g\gamma}$ corresponds to conjugate Γ_n by γ , but the conjugate is again Γ_n , since the subgroups Γ_n are normal.

We also have to prove that if $[s_e\Gamma_{\sigma_{p^e}}]$ is contained in $[s_{e-1}\Gamma_{\sigma_{p^{e-1}}}]$, where $s_e, s_{e-1} \in \Gamma, e \geq 1$ then $A_{s_e\sigma_{p^e}} \subseteq A_{s_{e-1}\sigma_{p^{e-1}}}$.

It is obvious that

$$A_{sg} = sA_g s^{-1}.$$

Hence to prove the inclusion it is sufficient to assume that s belongs to $\Gamma_{\sigma_{p^{e-1}}}$ and to prove that $sA_{\sigma_{p^e}}s^{-1} \subseteq A_{\sigma_{p^{e-1}}}$. But if $s \in \Gamma_{\sigma_{p^{e-1}}}$ then $s\sigma_{p^e} = \sigma_{p^e}\theta'$ for some θ' in Γ and hence

$$\begin{aligned} s\sigma_{p^{e-1}}\Gamma_n(\sigma_{p^{e-1}})^{-1}s^{-1} \cap \Gamma_n &= \sigma_{p^{e-1}}\theta'\Gamma_n(\theta')^{-1}(\sigma_{p^{e-1}})^{-1} \cap \Gamma_n = \\ &= \sigma_{p^{e-1}}\Gamma_n(\sigma_{p^{e-1}})^{-1} \cap \Gamma_n. \end{aligned}$$

Thus $sA_{\sigma_{p^{e-1}}}s^{-1} = A_{\sigma_{p^{e-1}}}$ and hence, since $A_{\sigma_{p^e}} \subseteq A_{\sigma_{p^{e-1}}}$ (by the choice we made for the groups Γ_n) it follows that $sA_{\sigma_{p^e}}s^{-1} \subseteq A_{\sigma_{p^{e-1}}}$.

Thus, as in the case of subgroups in \mathcal{MS} , any infinite intersection of subgroups in \mathcal{MS}_ω , when intersected with $\mathcal{C}_\omega((\Gamma_n)_n)$, will contain a reunion of infinite intersections of the form

$$(*) \quad \mathcal{C}_\omega((\Gamma_n)_n) \cap A_{s_1\sigma_p} \cap \dots \cap A_{s_e\sigma_{p^e}} \cap \dots$$

where $[s_1\Gamma_\sigma] \supseteq [s_2\Gamma_{p^2}] \supseteq \dots \supseteq [s_e\Gamma_{\sigma_{p^e}}]$, and $s_e \in \Gamma, e \geq 1$.

Again this will depend only on a coset of a point in $P^1(\mathbf{Z}_p^2)$ that in turn determines a coset of K/K_∞ . By compacity any infinite intersection corresponds to $k \in K/K_\infty$. We denote the infinite intersection in formula (*) corresponding an element $(y, t) \in P^1(\mathbf{Z}_p^2)$ (which in turns corresponds to $[s_1\Gamma_\sigma] \supseteq [s_2\Gamma_{\sigma_{p^2}}] \supseteq \dots \supseteq [s_e\Gamma_{\sigma_{p^e}}]$) by $K_{(y,t)}^\omega$.

We will verify the same property of intersection for this class of subgroups as the one holding for the for subgroups in \mathcal{MS} . We check that $K_{(y_1,t_1)}^\omega \cap K_{(y_2,t_2)}^\omega \cap K_{(y_3,t_3)}^\omega$ is contained in the kernel of the morphism from $\mathcal{MS}_\omega \cap \mathcal{C}_\omega((\Gamma_n))$ onto $K = \text{PSL}(2, \mathbf{Z}_p)$.

Indeed to check this we may assume that $(y_1, t_1) = (0, 1)$ in $P^1(\mathbf{Z}_p^2)$. Thus assume representatives for $(y_2, t_2), (y_3, t_3)$ are $\begin{pmatrix} x_2 & y_2 \\ z_2 & t_2 \end{pmatrix}$ and $\begin{pmatrix} x_3 & y_3 \\ z_3 & t_3 \end{pmatrix}$ and

$$K_{(y_1,t_1)}^\omega = K_{(0,1)}^\omega = \mathcal{C}_\omega((\Gamma_n)) \cap \bigcap_{e \geq 1} \mathcal{C}_\omega(\Gamma_n \cap \sigma_{p^e} \Gamma_n \sigma_{p^e}^{-1}).$$

Assume that $[s_e^j \Gamma_{\sigma_{p^e}}]$ are the decreasing sequence of cosets that determine $K_{(y_j,t_j)}^\omega$, and thus we may assume $s_e^j = \begin{pmatrix} x_e^j & y_e^j \\ z_e^j & t_e^j \end{pmatrix}$, $e \geq 1, j = 1, 2$, where the sequence (y_e^j, t_e^j) in $P^1(\mathbb{Z}_{p^e}^2)$ represents (y_j, t_j) in $P^1(\mathbf{Z}_p^2)$.

Then $K_{0,1}^\omega \cap K_{(y_j,t_j)}^\omega$, for $j = 1, 2$, by the same computations that we have performed for the subgroups of $\text{PSL}(2, \mathbf{Z}_p)$, consists of the group of sequences:

$$\left\{ \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \in \mathcal{C}_\omega((\Gamma_n)_n) \mid b_n \equiv 0, y_j^e t_j^e (a_n - d_n) \equiv (t_j^e)^2, \text{ mod } p^{s_n+e} \right\}$$

Because $(y_e^j, t_e^j)_e, j = 1, 2$ in the p -adic completion correspond to different elements in (y_j, t_j) in $P^1(\mathbf{Z}_p^2)$, the triple intersection will be contained

in

$$\left\{ \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \in \mathcal{C}_\omega((\Gamma_n)_n) \mid a_n \equiv d_n \ c_n \equiv 0 \pmod{p^{s_n+e-f}}, b_n \equiv 0 \pmod{p^{s_n+e}} \right\},$$

where f depends on which power of p divides (y_j, t_j) .

Replacing e by $e + f$, when necessary, this is contained in the required kernel.

To complete the proof we note that because of this argument, the only non-trivial intersections of subgroups in $\mathcal{MS}_\omega \cap K$ are the intersections

$$K_{(y_1, t_1)}^\omega \cap K_{(y_2, t_2)}^\omega$$

which may also be intersected by finite intersection of the form

$$\bigcap_{i=1}^r \mathcal{C}_\omega((\Gamma_n \cap g_i \Gamma_n g_i^{-1})_n),$$

where g_1, g_2, \dots, g_r belongs to G .

□

Using the terminology introduced above and using the fact that all subgroups in $(\Gamma_n)_n$ are normal, and since \tilde{F} is itself contained in $\mathcal{C}_\omega((\Gamma_n)_n)$, it obviously follows, using the conjugacy action Δ of G on $\mathcal{Y}_{\tilde{A}}$, that

$$\Delta(\Gamma)\tilde{F} \subseteq \mathcal{C}_\omega((\Gamma_n)).$$

Recall that s_i^e are the coset representatives for $\Gamma_{\sigma_{p^e}}$ in Γ , for $e \geq 1$. Then the set $\Gamma\tilde{F}_\infty$ (where Γ acts by conjugation) is contained in

$$\bigcap_{e \geq 1} \left(\bigcup_i \mathcal{C}_\omega \left(s_i^e \sigma_{p^e} \Gamma_n (\sigma_{p^e})^{-1} (s_i^e)^{-1} \cap \Gamma_n \right) \right).$$

The set in the above formula is the preimage, through Φ_ω , of the set

$$S_\infty = \bigcap_e \left(\bigcup_i K_{s_i^e \sigma_{p^e}} \right).$$

Clearly:

$$S_\infty = \bigcup_{(y,t) \in P^1(\mathbf{Z}_p^2)} K_{(y,t)} = \bigcup_{s \in K/K_\infty} sK_\infty s^{-1}.$$

The reunion above is not disjoint. Because of Lemma 32 it follows that the set S_∞ is, excluding the identity element of K , a disjoint reunion of sets of the form

$$K_{(y_1,t_1)} \cap K_{(y_2,t_2)} \cap \bigcap_{i=1}^r K_{g_i},$$

where $(y_1, t_1), (y_2, t_2)$ are distinct points in $P^1(\mathbf{Z}_p^2)$ and g_1, g_2, \dots, g_r belong to G . A similar statement holds true, modulo the kernel set, in \mathcal{MS}_ω (modulo the kernel).

Recall that K_∞ is the closure of $\Gamma_\infty = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. We prove bellow that the Koopman representation state corresponding to the vector associated to the characteristic function of the set \tilde{F}_∞ decomposes as a direct integral of states supported on

$$\bigcup_{s \in K/K_\infty} \Phi_\omega(sK_\infty s^{-1}).$$

Proposition 35. *The dynamics of the subsets of \tilde{F}_∞ under the action of G is determined, excluding the the intersection of \tilde{F}_∞ with the set defined in formula (25) by the dynamics of the (conjugation) action of G on $\mathcal{MS} \cap K$. We have:*

(i) *The only possible intersections of subgroups in $\mathcal{MS} \cap K$ that, under the action of G , reintersect nontrivially K , excluding the identity element e of K , are the sets*

$$K_{(y,t)} \cap K_{(y_1,t_1)} \cap K_{g_1} \cap \dots \cap K_{g_r},$$

where $(y, t), (y_1, t_1)$ are distinct elements in $P^1(\mathbf{Z}_p^2)$ and g_1, g_2, \dots, g_r are elements in G .

(ii) $g(K_{(y,t)})g^{-1} \cap K$ is non-trivial if and only if (y,t) corresponds to $s \in K/K_\infty$ with the property that $K_{y,t} = sK_\infty s^{-1}$. In this case necessary g is of the form $s\sigma_{p^e}$ for some σ_{p^e} , $e \geq 1$.

(iii) $\sigma_p(K_{(0,1)} \cap K_{(y,t)})$ is $K_{0,1} \cap K_{y,pt} \cap \sigma_p(K_{0,1})$.

Proof. The only part of the statement that was not yet proved is the statement about $g(K_{y,t})g^{-1} \cap K$.

To prove this we may assume that $(y,t) = (0,1)$ in $P^1(\mathbf{Z}_p^2)$ and hence we are analyzing the set

$$L_\infty = K \cap g(K \cap \sigma_p K \sigma_p^{-1} \cap \dots \cap \sigma_{p^e} K (\sigma_{p^e})^{-1} \cap \dots) g^{-1}.$$

But, unless g is of the form $s\sigma_{p^e}$ for some $e \geq 1$, the intersection is trivial. In the non-trivial case the intersection is

$$L_\infty = s\sigma_{p^e}(K_\infty)(\sigma_{p^e})^{-1}s^{-1}.$$

The last computation is trivial. □

Corollary 36. *Except for the subset of \tilde{F}_∞ defined by the formula*

$$(26) \quad A_{\infty,\infty} = \bigcup_{e>1} \{(a_n) \in A_n \mid a_n \in \Gamma_{p^{s_n+e}} \text{ } \omega - \text{eventually}\},$$

the rest of \tilde{F}_∞ corresponds to a central projection in the Koopman representation of $C^(G \ltimes L^\infty(\mathcal{Y}_{\tilde{A}}, \nu_\alpha))$, verifying the conditions of Theorem 27.*

Proof. Because of (i), (ii) in the preceding proposition, it remains to analyze the dynamics of the action by conjugation of G on the subset of S_∞ . Because of (iii) in the preceding proposition, the only part of S_∞ that, under the action of G that intersects again S_∞ , by the action by elements in the group G , is

$$\bigcup_{\gamma \in \Gamma/K_\infty \cap \Gamma} \gamma K_\infty \gamma^{-1}.$$

In this case the only movements by G that bring back pieces of $F_{\infty,1}$ are those implemented by $K_\infty \cap G$, which is an amenable group. \square

Formula (26) may also be rewritten as

$$A_{\infty,\infty} = \bigcup_{\substack{r_n \in {}^*\mathbb{N} \\ \lim_{n \rightarrow \omega} (r_n - s_n) = \infty}} \{a_n \in A_n \mid a_n \in \Gamma_{p^{r_n}} \text{ eventually, with respect } \omega\}$$

Lemma 37. Define $f : {}^*\mathbb{N} \rightarrow [0, 1]$ by the formula

$$f((r_n)) = \lim_{n \rightarrow \omega} \frac{\text{card}(A_n \cap \Gamma_{p^{r_n}})}{\text{card } A_n}.$$

By definition $f((s_n)) = 1$. Clearly f is decreasing. Then

$$\inf_{r \in {}^*\mathbb{N}} f(r) = 0.$$

Proof. Assume to get a contradiction that $\inf_{r \in {}^*\mathbb{N}} f(r)$ is not zero.

Then we would find $(\gamma_n) \in \mathcal{C}_\omega(\tilde{A}_n)$ such that (γ_n) belongs ω -eventually to $(\Gamma_{p^{r_n}}^0)$ for all $(r_n) \in {}^*\mathbb{N}$.

$$\text{Write } \gamma_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}.$$

Then the above shows that $(\forall) r_n \in {}^*\mathbb{N}$ we have b_n is divisible by p^{r_n} , and hence $b_n \geq r_n$ eventually for all $(r_n) \in {}^*\mathbb{N}$.

But this is impossible. \square

Proof of Theorem 27. Fix $\varepsilon > 0$ and (r_n^0) in ${}^*\mathbb{N}$ such that $f((r_n^0)) < \varepsilon$. Then almost all the mass, with the exception of a set of measure less than ε , of $\mathcal{C}_\omega(\tilde{A}_n)$ is concentrated in $((\Gamma_{p^{r_n}}^0))$.

We apply again the construction in this section to the decreasing sequence of groups $(\Gamma_{p^{r_n^0}}^0)_{n \in \mathbb{N}}$. Then the argument shows that except for the pieces of $\mathcal{C}_\omega(\tilde{A}_n)$ concentrated in an even smaller decreasing sequence of subgroups $(\Gamma^0(q_n))_{n \in \mathbb{N}}$, $q_n - r_n^0 \rightarrow \infty$, the rest of $\mathcal{C}_\omega(\tilde{A}_n)$ generated states on G that in turn correspond to Koopman states on $C^*(G \times G^{\text{op}} \ltimes L^\infty(\mathcal{Y}_{A^0}, \nu))$,

such that the Koopman state corresponding to the characteristic function of $\mathcal{C}_\omega((A_n^0))$ is a limit of double positive states with support contained in a finite reunion of double cosets of Γ .

But by the choice of r_n^0 , the remaining part

$$\mathcal{C}_\omega((\tilde{A}_n \cap \Gamma^0(q_n))_{n \in \mathbb{N}}),$$

will have mass $< \varepsilon$.

□

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